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AN HISTORICAL AND CRITICAL
STUDY OF THE FUNDAMENTAL
LEMMA IN THE CALCULUS OF
VARIATIONS

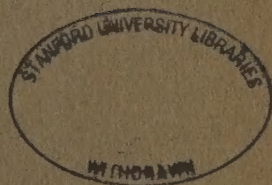
A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

By
ALINE HUKÉ

L. C.

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This dissertation is one of a group of five submitted during the year 1930 to the Department of Mathematics of the University of Chicago and concerned with topics in the theory of the Calculus of Variations. The group as a whole has been published in book form by the University of Chicago Press under the title *Contributions to the Calculus of Variations, 1930*.

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CHAPTER I

INTUITIVE DEDUCTIONS OF THE CONCLUSION OF THE FUNDAMENTAL LEMMA

1. Introduction. For nearly a century after Lagrange introduced his method of variations, the theory of maxima and minima of integrals was treated as an application of the theory of δ processes. For an integral

$$\int_a^b V(x, y, y', \dots, y^{(m)}) dx$$

the first variation in the δ notation of Lagrange is

$$\delta \int_a^b V dx = \int_a^b M \delta y dx + W \Big|_a^b,$$

where

$$M(x) \equiv N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \dots + (-1)^m \frac{d^m R}{dx^m},$$

$$W(x) \equiv \left(P - \frac{dQ}{dx} + \dots + (-1)^{m-1} \frac{d^{m-1} R}{dx^{m-1}} \right) \delta y$$

$$+ \left(Q - \dots + (-1)^{m-2} \frac{d^{m-2} R}{dx^{m-2}} \right) \frac{d \delta y}{dx}$$

$$+ \dots + R \frac{d^{m-1} \delta y}{dx^{m-1}},$$

$$N \equiv \frac{\partial V}{\partial y}, P \equiv \frac{\partial V}{\partial y'}, Q \equiv \frac{\partial V}{\partial y''}, \dots, R \equiv \frac{\partial V}{\partial y^{(m)}}.$$

The fundamental lemma for this case, as phrased more or less explicitly by the early writers, is the following: if

$$(1.1) \quad \int_a^b M \delta y dx + W \Big|_a^b = 0$$

for arbitrary variations $y(x)$, it follows that $M \equiv 0$ and the coefficient of δy and its derivations in W vanish separately.

The limits of integration have been indicated here after the manner of Ohm, 1825 (36)¹, Stegmann, 1854 (37), and others, but in the writings of Lagrange, 1806 (3), Euler, 1793 (2), Boucharlat, 1830 (10), and the earliest authors, no limits of integration appear. Lack of clearness in the statement of the fundamental lemma is probably partly due to this. For a fixed end point problem in the calculus of variations, the values of δy and its derivatives in the second term of the left member of equations (1.1) vanish at both end points and this term drops out. For $m = 1$ the lemma stated above then becomes the following: if

$$(1.2) \quad \int_a^b M \delta y dx = 0$$

for all arbitrary functions $\delta y(x)$ vanishing at $x = a$, $x = b$, then $M \equiv 0$. It is this last formulation of the lemma that is now generally known as the fundamental lemma of the calculus of variations (31, p. 25).

¹) Hereafter, numerals with or without an accompanying page number in a parenthesis will refer to the bibliography listed at the end of this paper.

To many of the early writers the condition $M \equiv 0$ seemed a self evident consequence of (1.1) . To others it seemed evident that each term of the left member of equation (1.1) must vanish separately and hence from the arbitrary character of δy it again seemed obvious that $M \equiv 0$ would follow from equation (1.2) holding for all arbitrary $\delta y(x)$.

In §2 of this chapter will be found a statement concerning the origin of the lemma in a letter of 1775 from Lagrange to Euler (1). Euler, 1793 (2), as Lagrange did at first, accepts the conclusion of the fundamental lemma as self evident. Lagrange, 1808 (4), in a later attempt to justify the conclusions of the lemma, merely assumes the lemma again. The remarks of others who disguise, though probably unconsciously, the fact that they really assume the lemma are also discussed in §2. In §3 attention is called to a statement which seems to contain an error in reasoning common to three of the early writers. In a somewhat different situation Brunacci, 1808 (11), also makes this error. Lecat (12) remarks that Moigno - Lindelöf, 1861 (13), assume the lemma as evident. In §4 a reference is given to a proof of the lemma by Moigno - Lindelöf. A list of those authors who obtain the conclusions of the fundamental lemma from equation (1.1) with no explanation whatsoever is also given in §4.

2. Remarks by Euler, Lagrange, Ostrogradsky, Jellett,

Carll. The earliest use of the lemma is found in a letter from Lagrange to Euler (1) August 12, 1755, where its conclusion is given as self evident. In his later investigations of problems involving the calculus of variations, based on the method of Lagrange, Euler, 1793 (2), also accepts the lemma as obvious.

Again in 1806, Lagrange (3, p. 438) obtains without proof the Euler - Lagrange equation, $M \equiv 0$, from the vanishing of the first variation; but he must have felt this somewhat inadequate, for in 1808 (4) the following argument is found. Let V be some function of x, y, y' having the primitive function $U(x)$ when $y = \phi(x)$, but having no formal primitive expressible in terms of x, y . Let y be represented by $\phi(x, i)$ such that when $i = 0, y = \phi(x)$. Then

$$\dot{U} \equiv \left. \frac{\partial U}{\partial i} \right|_{i=0} = 0$$

is a necessary condition for U to be a maximum or minimum when $i = 0$. But $\dot{U}' = \dot{V}$, so that a necessary condition for a maximum or minimum is that the primitive function of $\dot{V}$ shall vanish independently of $\dot{y}$. From the following expression for $\dot{V}$,

$$\dot{V} = (N - P' + Q'' - \dots) \dot{y} + (P\dot{y})' - (Q'\dot{y})' + (Q\dot{y})' - (R'\dot{y}')' + \dots$$

it is seen that all the terms except the first are exact derivatives, i. e. are such that their primitives are known and determined whatever y may be; while the first term can have a primitive function only if particular values are

assigned to y . Since this variation is to remain undetermined, the primitive function of V can vanish only if the coefficient of y occurring in the expression for $\dot{V}$ vanishes, which gives

$$N - dP/dx + d^2Q/dx^2 - \dots = 0.$$

This last statement of Lagrange is merely the fundamental lemma disguised by the introduction of the word, *primitive*.

Ostrogradsky, 1850 (6), also argues from the integrability of δV in a similar manner.

The superficial character of Lagrange's proof becomes more apparent in the modification given by Jellet, 1850 (7), for which the original argument by Lagrange very probably may have been the source. Jellet is concerned with equation (1-2) and writes, "Now it is known that as a result of the process of integration depends in general on the form of the function to be integrated, it is impossible to determine the value of such an expression as

$$\int_{x_0}^{x_1} \phi \delta y \, dx$$

without at least restricting the generality of the function

Hence it is evident that it is impossible to satisfy

$$\int_{x_0}^{x_1} \phi \delta y \, dx = 0$$

without either restricting the generality of the form of δy or making $\phi = 0$. As the former supposition is inadmissible, it is plain that we must have $\phi = 0 \dots$ "

Like Lagrange, Carll, 1881 (5), also disguises the fact that he has really assumed the fundamental lemma. He assumes that each term of the left member of (1.1) vanishes separately, for otherwise, since δy is an arbitrary function,

the integral of an arbitrary function would be expressed in terms of the limiting values of itself and a certain number of its differential coefficients which is "manifestly untrue." Carll then argues correctly that since $\mathcal{W} \Big|_a^b$ is linear in $\delta y \Big|_b^b$, $\delta y \Big|_a^a$, $\delta y' \Big|_b^b$, $\delta y' \Big|_a^a$, ... and since these are independent, each coefficient must vanish separately. However, he merely asserts that equation (1.2) cannot hold unless M vanishes.

3. The Methods of Bossut, Gergonne, Bourcharlat, Brunacci. Three authors state equation (1.2) as a result of the vanishing of the first variation without proof. These three, Bossut, 1798 (8), Gergonne, 1822 (9), and Boucharlat, 1830 (10), indicate no limits of integration, which is perhaps the reason that they disregard the fact that they are dealing with definite integrals and conclude, "Comme la différentielle d'une fonction égale à zéro est nulle aussi bien que cette fonction, ... , on peut ôter la signe d'intégration, ce qui nous donnera $M \delta y \, dx = 0$."

Brunacci, 1808 (11), obtains the Euler-Lagrange equation as a necessary consequence of equation (1.1) without any explanation. But he also makes an argument from the integrability of the first variation. He does not use the Lagrange integration by parts, but for the first variation of

$$U \equiv \int_a^b V(x, y, p) \, dx,$$

where $p = dy/dx$, considers

$$(1.3) \quad \delta \int V(x, y, p) \, dx = \int \left(\delta y \, dV/dy + \frac{d\delta y}{dx} \frac{dV}{dp} \right) dx.$$

Since this must vanish for all $\delta y(x)$, it must be integrable.

It will be noted that no limits of integration have been indicated. Brunacci then makes the same mistake made by the authors mentioned above; he disregards the fact that he is dealing with definite integrals and writes

$$(1.4) \quad \int (\delta y N + \frac{d\delta y}{dx}) dx = \alpha + \beta \delta y$$

where α and β are functions of x and p to be determined. The left member of (1.4) is treated as a definite integral and differentiated with respect to x . Equating coefficients of δy and $d\delta y/dx$ on both sides of the resulting equation gives the Euler - Lagrange equation, $M \equiv 0$.

4. A correction, and a list of authors who left the justification of the lemma to the intuition of the reader.

Lebat, 1913 (12), remarks that Moigno - Lindelöf assume the lemma as evident. However, one finds in the discussion of multiple integrals by Moigno - Lindelöf, 1861 (13), a proof of the lemma in which the form chosen for δy is due to Sarrus. The proof of Sarrus will be discussed later in Chapter II.

References to authors who leave the justification of the equation, $M \equiv 0$, as a consequence of equation (1.1) to the intuition of the reader are references 14 to 30 given on pp. 96 - 98 below.

CHAPTER II

THE PERIOD OF EARLY PROOFS AND ATTEMPTED PROOFS (1823 - 1870).

1. Introduction. Bolza, 1909 (31), and Dresden, 1926 (32), have attributed the first successful proof of the fundamental lemma to Stegman (1854). At least seven writers attempted proofs before that time. The earliest consciously attempted proof was that of Dirksen, 1823 (33). It was the forerunner of all subsequent proofs of the group to be considered in this chapter. It will be seen that, even with sufficient continuity conditions assumed, Dirksen's proof would not hold for the simplest fixed end point problem of the calculus of variations; but his contribution, that of first realizing the need for a proof and of suggesting a method must not be overlooked.

Many writers attempted proofs when once the necessity was realized. Many of these proofs are based on vague statements of the lemma, or involved inaccuracies in reasoning, or both. If Cauchy, 1831 (38), had exhibited a function δy satisfying the conditions which he states only in general terms, one might feel that his was the earliest satisfactory proof. With this lacking, the earliest proof which the writer has found and which seems successful in the sense that it

could be phrased to satisfy modern ideas of rigor, is that of Sarrus, 1848 (46).

In this chapter, brief descriptions of these proofs and others related to them are given.

2. The proof of Dirksen and related proofs. The earliest attempt to give what might justifiably be called a proof, since it served as a model for later proofs, is that of Dirksen, 1823 (33). Dirksen's argument is as follows, although the notation used is that introduced on page I of this paper.

Since equation (1.1) is independent of the special form of δy , it is satisfied for

$$(2.1) \quad \delta y = (x-a)^{m+p} (x-b)^{m+q},$$

where p, q are arbitrary positive integers. For such a δy , $W|_a^b$ vanishes. Hence the first term of the left member of equation (1.1) also vanishes and equation (1.1) reduces to the two equations

$$(1.2) \quad \int_a^b M \delta y dx = 0,$$

$$(2.2) \quad W|_a^b = 0.$$

To authors previously discussed, the vanishing of M seemed a self evident consequence of equation (1.2). Dirksen realized the need of proving this. He arrives at an absurdity by supposing that M does not vanish on the whole interval (a, b) . For

$$(2.3) \quad \delta y \equiv f(x)/M(x),$$

where $f(x)$ is arbitrary, equation (1.2) becomes

$$\int_a^b f(x) dx = 0;$$

but this is absurd since $f(x)$ is arbitrary.

Hence

$$(2.4) \quad M=0, \quad W \Big|_a^b = 0$$

necessarily follow from equation (1.1)

The first objection to Dirksen's proof is his conclusion that, since $W \Big|_a^b$ vanishes for δy given by (2.1), relation (1.2) then holds for all δy . A second objection is that with no properties of $f(x)$ specified, the form (2.3) used for δy would not satisfy the end conditions imposed upon δy by the fixed end point problem, for example. A third objection is that with M occurring in the function chosen for δy continuity properties assumed for δy would necessitate continuity properties of higher order for the minimizing arc. If (1.2) is to hold for δy of class C' , then $y(x)$ must be of class C''' for a calculus of variations problem involving first derivatives in order that $M(x)$ in the expression (2.3) may have a continuous derivative. Still another objection is that Dirksen shows only that the case when M is different from zero on the whole interval (a,b) is impossible, while it has been shown later by others that M cannot be different from zero even at a single point of the interval at which M is continuous.

Dirksen's proof was adopted by Bordoni, 1831 (34), and Klügel, 1831 (35).

Ohm, 1825 (36), removed the first objection to Dirksen's proof, recognizing the weakness in Dirksen's deduction of equation (1.2) and (2.2). He shows that if the coefficients

$$P \Big|_a^b, \quad Q \Big|_a^b, \quad \dots, \quad P \Big|_a^2, \quad Q \Big|_a^2, \quad \dots$$

are not all zero there is a particular δy for which (1.1) cannot hold. Hence all the coefficients $P|_b^b, Q|_b^b, \dots, P|_a^a, Q|_a^a, \dots$ must be zero and (1.2) holds for every δy . To show this, Ohm replaces δy by an arbitrary function of x with an indefinite number of undetermined constants. Then the integral in (1.1) will be a function of a and b and these constants. Hence one of them can be chosen so that equation (1.2) is satisfied. Since then equation (2.2) must be satisfied for this δy which still contains an indefinite number of undetermined constants, the coefficients $P|_a^a, Q|_a^a$, etc. must vanish separately. This follows from a lemma proved in an earlier section (§6: §87).

To obtain the first of equations (2.4), Ohm uses the function (2.3) with $f(x) \equiv 1$.

Stegmann, 1854 (37), adopts Ohm's proof. In addition, in the case when δy and a certain number of its derivatives must vanish at $x = a$, $x = b$, which happens for a fixed end point problem, Stegmann chooses

$$\delta y = (x-a)^2 (b-x)^2 / M(x).$$

In Stegmann's proof the first two objections to Dirksen's proof have been removed. With sufficient continuity properties assumed for the minimizing arc, and a slight modification making use of the continuity of M to remove the last objection, Stegmann's proof is valid for the fixed end point problem; and Ohm's proof is valid for the variable end point problem, where at the end points the x coordinate is fixed but y and its derivatives are allowed to vary.

3. Cauchy's suggestion. Cauchy, 1831 (38), considers a fixed end point problem and gives a suggestion for obtaining the Euler-Lagrange equation, $M \equiv 0$, from condition (1.2) when

$\delta y(x)$ is an arbitrary function vanishing at $x = a$, $x = b$. His argument is as follows. If M does not vanish everywhere on the interval (a, b) the arbitrary function δy can be chosen so that it always has the same sign as M . Hence the integrand of the integral in equation (1.2) will always be positive or zero without being identically zero. This gives the integral a positive value, contradicting equation (1.3). However, no explicit formula for δy satisfying these conditions is given.

Sturm, 1859 (39), Serret, 1868 (40), Weierstrass, 1879 (41), Duhamel, 1887 (42), and Laurent, 1890 (42), have adopted Cauchy's argument.

4. Crelle's discussion and its suggestiveness. An article by Crelle, 1833 (44), is of interest because of his unusual notation and treatment of the definite integral. He makes the following explanation, "Die Integrale, welche nichts anders sind, als Differentiale niedrigerer Ordnung, werden nicht wie gewöhnlich durch das willkürliche, fremde Zeichen $\int$ sondern, weil sie gleichfalls Differentiale niedrigerer Ordnungen sind, ebenfalls durch d , aber mit negativem Exponenten bezeichnet." For example, if

$$v = f\left(x, y, \frac{dy}{x}, \dots, \frac{d^n y}{x^n}\right),$$

then the integral of v with respect to x is denoted by

$$u = \frac{d^{-1}v}{x} = F\left(x, y, \frac{d}{x}y, \dots, \frac{d^{n-1}}{x^{n-1}}y\right).$$

Equation (1.2) in the notation of Crelle is

$$(2.5) \quad \frac{d^{-1}(\overset{1}{y})}{x} (\gamma_0 \frac{d}{t} y) - \frac{d^{-1}(\overset{2}{y})}{x} (\gamma_0 \frac{d}{t} y) = 0$$

where γ_0 1,2 correspond to M, b, a of page 1 respectively, and where the y's with superscripts indicate that the expressions in parentheses following them are to be evaluated at the points 1 and 2. Here

$$(2.6) \quad \delta y \equiv \left(\frac{d}{t} y \right)_{t=t_0}, \quad y = \phi(x, t).$$

Crelle, however, does not indicate this last relation. He interprets equation (2.5) as meaning that the integral of $\gamma_0 \frac{d}{t} y$ has the same value for $\overset{1}{y}$ as for another value $\overset{2}{y}$ of y, which is possible only for special functions y of t. Every different function y will give the integral a different value; but y is arbitrary and hence γ_0 must vanish. This doubtless sounded plausible to Crelle but amounts to nothing more than assuming the lemma.

Crelle, however, makes an alternative proof. The function y is expressed as a power series in x with coefficients arbitrary functions of a parameter t. γ_0 is also expanded in such a series. This done, equation (2.5) gives

$$e_0 + e_1 \overset{1}{x} + e_2 \overset{1}{x}^2 + \dots = e_0 + e_1 \overset{2}{x} + e_2 \overset{2}{x}^2 + \dots$$

which, Crelle says, holds for fixed $\overset{1}{x}$, $\overset{2}{x}$ and all arbitrary functions $e_0, e_1, \dots$ of t, which is impossible unless $\gamma_0 = 0$.

That the $e_0, e_1, \dots$ are not wholly arbitrary but are subject to convergence properties in order that the differentiation, multiplication, and integration of these series

term by term may be justified and that the coefficients $a_0, a_1, a_2, \dots$, in the expansion of y must be such that $\frac{d}{dx} y$ vanishes with a certain number of its derivatives at the end points apparently did not occur to Crelle.

However, Crelle's method is suggestive of a method of proof by means of Legendre polynomials. Schauder, 1928 (45), has remarked that Du Bois Reymond's lemma can be proved by means of Legendre polynomials. The statement is also true for the fundamental lemma, although the writer has seen no proof in the literature by this method. It is easily made in a manner analogous to that of Schauder for double integrals.

5. The first complete proof by Sarrus in 1848. Although the purpose of the prize memoir of Sarrus, 1848 (46), was the investigation of the maxima and minima of multiple integrals, the results apply to simple integrals as a special case. In order to show that

$$(2.7) \quad M(x, y, z, \dots) = 0$$

if

$$(2.8) \quad \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \dots M \delta u \dots dz dy dx = 0$$

for all $\delta u(x, y, z, \dots)$ vanishing at the limits of integration, Sarrus chooses

$$(2.9) \quad \delta u = M \omega^{2n}$$

where

$$\omega = (x - x_1)(x - x_2)(y - y_1)(y - y_2)(z - z_1)(z - z_2) \dots$$

and n is some positive integer. The function δu thus defined when substituted in (2.8) gives

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \dots (M \omega^{2n})^2 \dots dz dy dx = 0.$$

Equation (2-7) follows at once from this.

For the simplest calculus of variations problem involving first derivatives, it is generally assumed that δy has at most a finite number of discontinuities; but for this problem the second derivative of the minimizing arc occurs in the expression for M and hence the use of the function (2-9) chosen for δu and of its derivatives assumes the existence of the third derivative of $y(x)$. Aside from this restriction, Sarrus' proof is an improvement over Stegmann's proof, for it shows that M cannot be different from zero at any point of (a, b) , while Stegmann's proof shows only that M cannot be different from zero on the whole interval (a, b) .

On page 7 of this paper attention has already been called to the fact that Moigno-Lindelöf, 1861 (13), adopted Sarrus' proof.

6. Proofs of Todhunter, Strauch, Dienger, and related proofs. Todhunter, 1861 (47), says that Strauch¹ has a curious notion concerning this lemma, asserting that it cannot be proved, but admitting that $M = 0$ does give solutions of the problem. Evidently Todhunter was not familiar with the

1) G. W. Strauch, Theorie und Anwendung des Sogenannten Variationscalcul's. 1849, vol. 2, preface pp. 25, 26.

second edition of this treatise by Strauch, 1854 (49), for there the lemma is proved by a reductio ad absurdum similar to the method of Stegmann. This is accomplished by assuming $M(x) \neq 0$ on the interval (a, b) and by setting

$$\delta y = \frac{1}{M(x)} \frac{d F(x)}{dx} ,$$

where $F(x)$ is an arbitrary function of x .

It is none too clear that Todhunter himself believes a proof necessary. Commenting on Strauch's "singular notion", he adds, "It may be safely asserted that the ordinary view of the necessity of the equation $M = 0$ is sound; supposing that whatever series of values δy can assume, it can also assume the corresponding series of values numerically equal but of opposite sign without changing the limiting values of the variations." Still another reference (48) seems to show that Todhunter's views concerning a proof of the lemma were not very clear. He says that if δy can be positive or negative conditions (2.4) are indispensable for a maximum or minimum and that "these statements are universally admitted to be true." Todhunter adds, however, that for one sided variations, it is no longer necessary that M should vanish.

Another proof similar in general character to those already mentioned in this chapter is that of Dienger, 1867 (50. He sets

$$(2.10) \quad \delta y = M \xi$$

where ξ is a function always positive interior to the interval of integration. Dienger does not say explicitly that f and a certain number of its derivatives vanish at $x = a$, $x = b$. However, this is stated explicitly by the following authors who give proofs by choosing functions of the form (2.10) for δy : Teixeira, 1892 (51), Hancock, 1897 (52), Jordan, 1915 (53), and Weierstrass, 1879 (?) (54).

Méray, 1877 (55), gives an incomplete proof. He gives an expression for δy which contains a wholly arbitrary function $\delta \eta$ and which takes care of the end conditions that y and a certain number of its derivatives may satisfy. With this substituted in (1.1), he obtains a similar expression,

$$A' + \int_a^b (x-a)^{h_2} (x-b)^{h_1} B' \delta \eta = 0.$$

Without giving any details he remarks that by the usual method of reasoning it follows that $E' = 0$.

CHAPTER III

MODERN PROOFS 1870 _____ .

1. Introduction. Modern proofs are distinguished by the fact that for the equation

$$(3.1) \quad \int_a^b M \delta y \, dx = 0,$$

the continuity properties of $M(x)$ and the class of functions δy , for which the equation holds, are clearly specified. Heine, 1870 (56), seems to be the first to have given such a proof for the case when $M(x)$ is continuous and changes sign a finite number of times on the interval (a, b) and when δy has a finite number of continuous derivatives. Not only were continuity properties of M and the class of functions δy for which (3.1) holds not specified in earlier proofs, but the proofs themselves required more restrictions than are necessary. All the proofs of Chapter II, with the exception of Cauchy's proof in which no explicit form for δy is given, make use of M in the auxiliary functions chosen for δy . It has already been noted that such proofs, when they are to be applied to the simplest calculus of variations problem involving only first derivatives, necessitate the existence of at least the third derivative of the minimizing arc, if δy is to have a derivative. On the other hand, the application of the proofs given in this chapter to a calculus of variations problem requires no further restrictions on the minimizing arc than are necessary for the transformation of the first

variation to the form (3.1) by the Lagrange integration by parts.

Du Bois Reymond, 1879 (58), gives proofs of the sort just described for the cases when M may have discontinuities and δy has continuous derivatives of all orders. Zermelo's proof, 1894 (61), shows that it is sufficient to restrict the class of functions δy for which (3.1) holds to analytic functions to obtain the vanishing of M as a consequence of equation (3.1). Landau, 1908 (70), presents two proofs for the case when M is continuous, $a = 0$, $b = 1$, and δy is of the form x^m , m being a positive integer. Rosenblatt, 1910 (71), gives a proof for the case when δy is a polynomial vanishing at $x = a$, $x = b$, Hobson, 1913 (72), extends Rosenblatt's proof to the case when M is summable and the class of functions δy for which equation (3.1) holds is such that δy is a finite polynomial vanishing with a finite number of its differential coefficients at $x = a$, $x = b$. He finds that M then vanishes almost everywhere on (a, b) .

2. The first modern proof by Heine in 1870. As has been remarked, Heine, 1870 (56), seems to have been the first to realize that δy is not entirely arbitrary, for he says concerning the lemma "In den meisten Lehrbüchern wird dieser Satz als selbstverständlich betrachtet, in dem man zu vergessern scheint, dass in den Fällen auf welche man ihn anwenden will, die sogenannte willkürliche Variation δy nicht willkürlich im gewöhnlichen Sinne, sondern der Bedingung unterworfen ist, es solle δy selbst und eine endliche Anzahl seiner aufeinander

folgenden Ableitungen nach x noch continuirlich bleiben."

In order to show that (1.2) is a consequence of (1.1), Heine sets

$$\delta y = (x - a)^m (x - b)^m (x - \alpha_1)(x - \alpha_2) \dots,$$

$x = \alpha_1, x = \alpha_2, x = \alpha_3, \dots$ being the points, finite in number, where M changes sign. With this choice of δy

$$W \int_a^b \delta y \, dx \text{ vanishes and (1.1) reduces to}$$

$$(3.2) \quad \int_a^b M \delta y \, dx = 0.$$

Since the integrand has constant sign on (a, b) for this choice of δy , it follows that M must vanish. We have seen that former proofs necessitated the existence and continuity of the third derivative of $y(x)$, if they were to be valid for the simplest problem where first derivatives only of y appear in the integrand of the integral to be minimized. The existence and continuity of the second derivative is sufficient for Heine's proof.

Pascal, 1899 (57), calls attention to the step taken by Heine in realizing that δy must satisfy certain continuity conditions and gives Heine's proof.

3. Proofs of Du Bois Reymond and Weierstrass, and related proofs. Du Bois Reymond, 1879 (58), proves the following: If equation (3.1) holds for all δy vanishing at $x = a, x = b$, and having first n differential coefficients continuous, then M is zero at each point of (a, b) where it is continuous on the left, or right, or both.

To prove this, the following δy is chosen:

$$\delta y = 0, \quad a \leq x \leq \alpha, \quad \beta < x \leq b, \quad a < \alpha < \beta < b,$$

$$\delta y = [(x - \alpha)(\beta - x)]^{n+1}, \quad \alpha \leq x \leq \beta.$$

This δy is never negative on (a, b) and is continuous with its first n differential coefficients. Equation (3.1) gives

$$0 = M(\xi) \int_{\alpha}^{\beta} \delta y \, dx, \quad \alpha \leq \xi \leq \beta,$$

provided that $M(x)$ is continuous on the interval (α, β) ,

or

$$M(\xi) = 0, \quad \alpha \leq \xi \leq \beta.$$

If α and β approach arbitrarily near some x' , the limit of $M(\xi)$ is zero. $M(\xi)$ is some unknown value lying between the smallest and largest functional values of $M(x)$ on $\alpha \leq x \leq \beta$. If now $M(x') - M(x' - \varepsilon)$ or $M(x') - M(x' + \varepsilon)$ or both approach zero with ε then $M(x')$ necessarily vanishes.

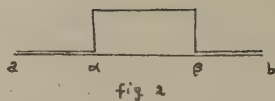
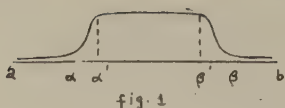
Du Bois Reymond also gives a proof for the case when (3.1) holds for all δy vanishing at $x = a$, $x = b$, having continuous derivatives of all orders. The proof is made by means of the following δy :

$$\delta y = 0, \quad a \leq x \leq \alpha, \quad \beta \leq x \leq b,$$

$$\delta y = 1, \quad \alpha' \leq x \leq \beta',$$

$$\delta y = \frac{1}{\pi} \left\{ \arctan \rho(x) + \frac{\pi}{2} \right\}, \quad \alpha < x < \alpha', \quad \beta' < x < \beta,$$

$$\rho(x) = \left(x - \frac{\alpha + \alpha'}{2} \right) e^{\frac{1}{(x - \alpha)(\alpha' - x)}}.$$



Since the intervals $\alpha\alpha'$, $\beta\beta'$ can be made arbitrarily small in comparison with $\alpha\beta$, δy shown in fig. 1 can be made to differ from that in fig. 2 by as little as is wished. It then follows easily that

$$\int_{\alpha}^{\beta} M \, dx = 0.$$

If M is continuous, α and β being arbitrary, it is seen that $M \equiv 0$ on (a, b) .

Attention has been called to a proof of Weierstrass, 1879 (?) (54), in which the form chosen for δy involves M ; but realizing that this proof does not hold if δy is differentiable, since M does not need to possess this property, Weierstrass (59) gives another proof. He attributes this proof to Du Bois Reymond, although the argument is somewhat different from that of the first proof of Du Bois Reymond just given. It is assumed that M is continuous on (a, b) , δy differentiable, and that δy vanishes at $x = a$, $x = b$. If M is different from zero at some point x' of (a, b) , e.g., $M(x') > 0$, since M is continuous, there exists an interval (x_3, x_4) containing x' in its interior and such that M is positive on $x_3 \leq x \leq x_4$.
For

$$\delta y = 0 \quad \text{on} \quad a \leq x \leq x_3, \quad x_4 \leq x \leq b,$$

(3.3)

$$\delta y = (x - x_3)^2 (x - x_4)^2 \quad \text{on} \quad x_3 \leq x \leq x_4,$$

the integral in (3.1) remains greater than zero contrary to hypothesis. Hence the assumption that $M(x') > 0$ is false.

The argument follows by parity if it is supposed that

$M < 0$ at some point of the interval.

With δy of the general form,

$$\delta y = 0 \quad \text{on} \quad a \leq x \leq x_3, \quad x_4 \leq x \leq b,$$

(3.4)

$$\delta y = (x - x_3)^k (x_4 - x)^k \quad \text{on} \quad x_3 \leq x \leq x_4,$$

some writers have followed Du Bois Reymond, using the Theorem of the Mean; while others have followed Weierstrass, arriving at a contradiction by showing that the δy used gives the integral in (3.1) a non zero value. It is generally assumed

that M is continuous on (a, b) or at most has a finite number of discontinuities. These authors are the following: Scheeffer, 1885 (60) who uses (3.4) with $k = 1$; Zermelo, 1894 (61),

$k =$ some positive integer; Kneser, 1900 (62), who uses (3.4)

with a small positive factor ε , $k = 3$, and who also later

in 1925 uses (3.4) with k any positive integer; Gernet, 1902

(63), $k = 3$; Hadamard, 1910 (64), $k = 1$; Vivanti, 1911 (65),

$k = 3$; Coursat, 1927 (66), $k =$ some positive integer.

4. Proofs of Zermelo and Kneser for δy analytic.

Following a suggestion which he says is due to Weierstrass, Zermelo, 1894 (67), proves the lemma for the case when M has at most a finite number of discontinuities by using a function for δy , expressed on the whole interval by a single analytic expression:

$$\delta y = (x-a)^m (b-a)^m e^{-\rho^2 (x-x_0)^2}$$

where ρ is a positive quantity independent of x . By means of this function a contradiction is obtained by assuming $M \neq 0$ at some point x_0 of (a, b) .

It is possible that the function chosen for δy may have been suggested also by Kirchhoff, 1891 (68), who called attention to the properties of such a function.

Kneser, 1925 (69), also derives the result $M \equiv 0$ when (3.1) holds for all analytic functions δy whose first k derivatives vanish at $x = a$, $x = b$. He makes use of the following auxiliary lemma:

Lemma. If $F(x)$ is continuous on $a \leq x \leq b$, then for $a < s < b$,

$$\lim_{c \rightarrow +\infty} \int_a^b F(x) e^{-c^2 (x-s)^2} dx = F(s) \sqrt{\pi}.$$

Kneser then chooses

$$\delta y = \left\{ \frac{(x-a)(b-x)}{(s-a)(b-s)} \right\}^k e^{-c^2(x-s)^2},$$

$$F(x) \equiv M \left\{ \frac{(x-a)(b-x)}{(s-a)(b-s)} \right\}^k$$

Now $F(s) = M(s)$, and from the auxiliary lemma,

$$\lim_{c \rightarrow +\infty} \int_a^b M \delta y \, dx = \sqrt{\pi} \, M(s).$$

But

$$\int_a^b M \delta y \, dx = 0.$$

Hence $M \equiv 0$.

These proofs show that to obtain $M \equiv 0$ as a consequence of equation (3.1) it is sufficient if the class of functions δy for which (3.1) holds is such that δy is analytic.

5. Proofs by Landau, Rosenblatt, Hobson when δy is a polynomial. Landau, 1908 (70), presents a proof due to Lerch, and one due to Stieltjes, of the following statement: if $M(x)$ is continuous on $0 \leq x \leq 1$ and if

$$(3.5) \quad \int_0^1 x^m M(x) \, dx = 0$$

for every $m \geq 0$, then $M \equiv 0$. Lerch's method is as follows.

If $M(x)$ is not identically zero, then

$$\int_0^1 |M| dx > 0$$

and a number can be chosen such that

$$(3.6) \quad 0 < \delta < \frac{\int_0^1 M^2 dx}{\int_0^1 |M| dx}.$$

Now a polynomial $G(x)$ can be chosen so that $M(x) = G(x) + \theta(x)\delta$ where $|\theta(x)| < 1$.

From (3.5)

$$\int_0^1 M G dx = 0$$

and hence

$$\int_0^1 [M(x)]^2 dx = \int_0^1 M(G + \theta\delta) dx < \delta \int_0^1 |M| dx$$

which contradicts the inequality (3.6).

Stieltjes' argument, briefly, is this: when ξ is some number between 0 and 1 and $M(\xi) > 0$, then

$$\int_0^1 [1 - (x - \xi)]^n M dx$$

is positive for sufficiently large n contradicting equation (3.5).

Rosenblatt, 1910 (71), also shows that to obtain $M \equiv 0$ from equation (3.1) it is sufficient if the class of functions ϕ for which (3.1) holds is the class of all polynomials vanishing at $x = a$, $x = b$. If $M \neq 0$ at some point x_0 in (a, b) then for

$$\delta y = 0, \quad a \leq x \leq x_0 - \delta, \quad x_0 + \delta \leq x \leq b,$$

$$\delta y = x - x_0 + \delta, \quad x_0 - \delta \leq x \leq x_0,$$

$$\delta y = x_0 + \delta - x, \quad x_0 \leq x \leq x_0 + \delta,$$

one finds that

$$(3.7) \quad \int_a^b M \delta y \, dx = k \neq 0.$$

Since a polynomial can be found vanishing at $x = a$, $x = b$, approximating δy sufficiently, for which equation (3.1) holds, equation (3.7) is impossible and hence M vanishes on (a, b) .

With few changes this method is extended by Hobson, 1913 (72), to prove the lemma in the following form: if $M(x)$ is summable on (a, b) and such that (3.1) holds for any finite polynomial δy which together with its first p differential coefficients vanishes at the end points of the interval, then $M(x)$ must vanish for every value of x except those belonging to some set of which the measure is zero. Also, M must vanish at every point at which it is continuous. In particular, if M is continuous on (α, β) which may be the whole or part of (a, b) , it must vanish everywhere on (α, β) .

6. The method of Schwarz. Hancock, 1904 (73), presents a proof due to Schwarz. It is assumed that M is continuous and δy is also continuous on (a, b) , vanishing at $x = a$, $x = b$. If M does not vanish at some point x' ($a < x' < b$), then from the continuity of M there exists an interval $(x' - d, x' + d)$ on

which M does not vanish. Schwarz considers

$$(3.8) \quad (y - 1 + x^2/d^2) y = \epsilon^2.$$

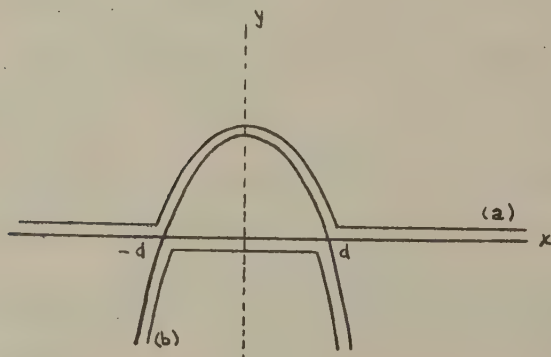
For ϵ sufficiently small this curve approaches the x axis and the parabola

$$y = 1 - x^2/d^2.$$

Then the curve defined by

$$(3.9) \quad y = (1 - x^2/d^2)/2 \pm \sqrt{(1 - x^2/d^2)/4 + \epsilon^2}$$

is represented by the branch (a) in the figure for the plus sign and by the branch (b) for the minus sign.



By choosing

$$\delta y = \frac{(x-a)^m}{(x'-a)^m} \frac{(b-x)^n}{(b-x')^n} \chi(x),$$

where

$$\chi(x) = \frac{1}{2} \left[1 - \frac{(x-x')^2}{d^2} \right] + \sqrt{\frac{1}{4} \left[1 - \frac{(x-x')^2}{d^2} \right] + \varepsilon^2},$$

Hancock shows that for ε sufficiently small the integral in (3.1) cannot vanish.

CHAPTER IV

MODIFICATIONS OF THE FUNDAMENTAL LEMMA

1. Introduction. For the transformation of the first variation of the integral

$$U = \int_a^b V(x, y, y', y^{(m)}) dx$$

by the Lagrange integration by parts, the existence at least of $y^{(2m)}$ must be assumed. Du Bois Reymond seems to be the first to have removed this restriction. The first variation of U before integration by parts is given by

$$\delta U = \int_a^b (V_y \delta y + V_{y'} \delta y' + \dots + V_{y^{(m)}} \delta y^{(m)}) dx$$

By what has since been called the Du Bois Reymond integration by parts, the vanishing of δU can be expressed by the following equation,

$$(4.1) \quad \int_a^b N(x) \eta^{(m)} dx = 0,$$

where $\eta \equiv \delta y$ and

$$N \equiv V_{y^{(m)}} - \int_a^x d\xi V_{y^{(m-1)}} + \dots + \frac{(-1)^m}{(m-1)!} \int_a^x (x-\xi)^{m-1} V_y d\xi,$$

the function $\eta(x)$ being subject to the end conditions

$$\eta(a) = \eta'(a) = \dots \eta^{(m-1)}(a) = 0$$

(4.2)

$$\eta(b) = \eta'(b) = \dots \eta^{(m-1)}(b) = 0.$$

Du Bois Reymond, 1879 (74), proved the following lemma:

Lemma. If N is continuous and if equation (4.1) holds

for all functions $n(x)$ satisfying conditions (4.2) and having continuous first m derivatives, then N is a polynomial of degree $m-1$.

Bolza (86: footnote 1 p. 28) and Lecat (12:21) with justice attribute the first proof of this lemma for the case $m = 1$ to Du Bois Reymond lemma. They also attribute the first general statement and proof of the lemma for m any positive integer to Zermelo. This is not well justified for there are two articles by Du Bois Reymond in volume 15 of the *Mathematische Annalen* to which Lecat and Bolza refer, and in the second article Du Bois Reymond proves the lemma for any m .

Proofs of Du Bois Reymond's lemma for any m have been given by Jacobsthal, 1910 (75), and Haar, 1922 (79), under the same hypotheses as those used by Du Bois Reymond. Zermelo, 1904 (78), assumes in his proofs that N and $y^{(m)}$ are integrable. Hobson, 1913 (80), has shown that the conclusion of the lemma holds almost everywhere on (a,b) if it is assumed that N is summable and $\eta(x)$ is in the class of all finite polynomials vanishing with their first $m-1$ differential coefficients at $x = a$, $x = b$. Szucs, 1915 (77), exhibits a specially selected function $\eta(x)$ by means of which he shows that the conclusion

of the lemma holds if it is assumed that $\eta(x)$ and its first s derivatives, $s \geq m$, vanish at $x = a$, $x = b$. It is explained in § 2 below how this same result can be obtained at once from Du Bois Reymond's proof by observation of the properties of the functions chosen for $\eta(x)$ in the proofs of the fundamental lemma given in the preceding chapter. By formulating conditions (4.2) as isoperimetric conditions, Du Bois Reymond expresses the hypotheses of the lemma of page 49 in a form to which the fundamental lemma of page 3 can be applied, $\eta^{(m)}(x)$ replacing $\delta y(x)$ and $N + c_0 + c_1 x + \dots + c_{m-1} x^{m-1}$ replacing M in equation (3.1). Because of this, Du Bois Reymond's proof has an advantage over other proofs. The results obtained, concerning restrictions on the class of functions to which $\delta y(x)$ may belong in order that $M = 0$ will follow from equation (3.1), can in fact be carried over and applied to $\eta^{(m)}(x)$. The proofs of other authors, with the exception of Hobson, only hold for $\eta^{(m)}$ either continuous or integrable.

Hobson, 1913 (81), has proved an extension of Du Bois Reymond's lemma. The function $\eta^{(m-1)}$ is replaced by $L(x, \eta(x))$, a linear, homogeneous, differential expression of order m . If

$$\int_a^b L(x, \eta(x)) N(x) dx = 0$$

holds for the class of functions $\eta(x)$ for which $\eta(x)$ is a polynomial and vanishes with its first $p-1$ differential coefficients at $x = a$, $x = b$, and if N is continuous, Hobson shows that N is a solution of the equation adjacent to

$L(x, \eta(x)) = 0$. This theorem contains Du Bois Reymond's lemma formally as a special case, but Hobson in his proof of the more general theorem makes use of Du Bois Reymond's lemma. Haar, 1922 (82), gives a proof which is a generalization of his own method of proving Du Bois Reymond's lemma in which it is assumed that N is continuous and $\eta(x)$ has continuous m th derivatives, vanishing with its first m derivatives at $x = a, x = b$.

Gauss, 1840 (85), used a method for a problem in potential theory which is closely related to the method used by Hilbert in his proof of Du Bois Reymond's lemma for the case $m = 1$. This relationship is discussed in § 5 below.

In most proofs of Du Bois Reymond's lemma for $m = 1$, it is assumed that N and $\eta'(x)$ have at most a finite number of discontinuities and that $\eta(x)$ vanishes at $x = a, x = b$. Of the proofs with these hypotheses, Jacobsthal's seems the nearest and most simple. In § 6 below attention is called to what seem to be inaccuracies in Kryloff's proof of the lemma, 1923 (83).

Bliss, 1908 (98), derives the Euler-Lagrange equations in the Du Bois Reymond form for the case $m = 1$ by obtaining first, necessary and sufficient conditions that

$$I \equiv \int_a^b [A(x, y(x)) + B(x, y(x)) y'(x)] dx$$

be independent of the path in some region. A theorem proved later by Razmadze, 1921 (99), is a special case of this theorem and follows immediately from it.

In the last section of this chapter will be found a somewhat artificial proof of Du Bois Reymond's lemma, a generalization by the author of a proof given by Jackson, 1922 (100), for the case $m = 2$.

2. Du Bois Reymond's lemma and proofs by Jacobsthal, Szűcs, Hadamard. In order to modify the restrictions (4.2), Du Bois Reymond, 1879 (74), considers the following isoperimetric problem. It is supposed that equation

$$(4.3) \quad \int_a^b N \mathcal{Y}^{(m)} dx = 0$$

holds for the class of functions $\mathcal{Y}^{(m)}(x)$ which are continuous and satisfy the equations

$$(4.4) \quad \int_a^b (b-x)^{m-1} \mathcal{Y}^{(m)} dx = 0, \dots, \int_a^b (b-x) \mathcal{Y}^{(m)} dx = 0, \int_a^b \mathcal{Y}^{(m)} dx = 0.$$

From the multiplier rule for isoperimetric problems there exist m constants $c_0, \dots, c_{m-1}$, such that the equation

$$(4.5) \quad \int_a^b \mathcal{Y}^{(m)} [c_0 + c_1(b-x) + \dots + c_{m-1}(b-x)^{m-1} + N] dx = 0$$

is satisfied not only for those functions $\mathcal{Y}^{(m)}$ satisfying (4.4) but for all continuous $\mathcal{Y}^{(m)}(x)$. Since $\mathcal{Y}^{(m)}$ is arbitrary it follows that N is a polynomial of degree $m-1$,

$$-N = c_0 + c_1(b-x) + \dots + c_{m-1}(b-x)^{m-1}.$$

To obtain this result, Du Bois Reymond gives in detail a method that he has used (58, p. 310) to prove the multiplier rule.

In order to prove the formulation (4.3) (4.4) equivalent to (4.1), (4.2) one should notice first that for functions $\eta(x)$ satisfying conditions (4.2), equations (4.4) are necessarily satisfied for $\zeta = \eta$. It remains to be shown that for any $\zeta(x)$ satisfying equation (4.4) there exists a function $\eta(x)$ such that $\zeta^{(m)} = \eta^{(m)}$ and such that equations (4.2) are satisfied. It can be verified that the following $\eta(x)$ is such a function:

$$\eta(x) \equiv \zeta(x) - \zeta(a) - \zeta'(a)(x-a) - \dots - \frac{\zeta^{(m-1)}(a)(x-a)^{m-1}}{(m-1)!}.$$

Jacobsthal's method, 1910 (75), is closely related that of Du Bois Reymond. It is easily seen from integration by parts that for $\eta(x)$ satisfying conditions (4.2),

$$\int_a^b \eta^{(m)} x^r dx = 0, \quad r = 0, 1, \dots, m-1.$$

This was perhaps suggested by equations (4.4). Then

$$(4.6) \quad \int_a^b \eta^{(m)} (c_0 + c_1 x + \dots + c_{m-1} x^{m-1} + c_m N) dx = 0$$

for any $m+1$ undetermined constants $c_0, c_1, \dots, c_m$.

The function $f(x)$ is defined by

$$f(x) \equiv \int_a^x \int_a^x \dots \int_a^x (c_0 + \dots + c_{m-1} x^{m-1} + c_m N) dx^m.$$

This function obviously satisfies the conditions

$f^{(r)}(a) = 0, \quad r = 0, \dots, m-1.$ These $m+1$ arbitrary

constants $c_0, \dots, c_m$ can now be determined so that they are not all zero and such that they satisfy the m homogeneous equations $f^{(r)}(b) = 0, (r = 0, \dots, m-1)$. With the constants $c_0, \dots, c_m$ thus determined, $f(x)$ satisfies the conditions on $\eta(x)$. For $\eta \equiv f(x)$, equation (4.6) becomes:

$$\int_a^b (c_0 + \dots + c_{m-1} x^{m-1} + c_m N)^2 dx = 0$$

and hence

$$c_0 + \dots + c_{m-1} x^{m-1} + c_m N = 0.$$

The constant $c_m \neq 0$ since $1, x, \dots, x^{m-1}$ are linearly independent.

This proof of Jacobsthal's only holds, however, for the class of functions $\eta(x)$ vanishing with their first $m-1$ derivatives at $x = a, x = b$ and having continuous m th derivatives. It would not hold, for example, for the class of analytic functions. From proofs of the fundamental lemma that have been given in Chapter III, it is clear from Du Bois Reymond's proof that to obtain the conclusion of the lemma from equation (4.5) it is sufficient if the class of functions for which (4.1) holds is the class of analytic functions or finite polynomials.

Szücs, 1915 (77), obtains the conclusion of Du Bois Reymond's lemma by assuming that (4.1) holds for all $\eta(x)$ having continuous first s derivatives such that $\eta(a) = \eta(b) = \eta'(a) = \eta'(b) = \dots = \eta^{(s)}(a) = \eta^{(s)}(b) = 0, s \geq m$. The proof is made by means of a specially constructed function $\eta(x)$.

From the functions chosen for $\eta(x)$ in proofs of the fundamental lemma (e. g. (3.4)), it is clear that $M = 0$ if (3.1) holds for all $\delta y(x)$ having a finite number of continuous derivatives for which $\delta y(a) = \delta y(b) = \dots = \delta y^{(s)}(a) = \delta y^{(s)}(b) = 0$. This applied to (4.5) with M replaced by $c_0 + c_1 x + \dots + c_{m-1} x^{m-1} + N$, $\delta y(x)$ by $\eta^{(m)}(x)$ gives Szucs result.

Hadamard, 1910 (76), also proves the lemma by formulating (4.2) as isoperimetric conditions.

3. Zermelo's proofs of Du Bois Reymond's lemma. Two proofs have been given by Zermelo, 1904 (78). The first makes use of an artificially constructed function for $\eta(x)$. The second is somewhat different. It is first shown that if

$$(4.7) \quad \int_a^b g(x) z \, dx = 0$$

where z is an integrable function of x and $g(x)$ is any polynomial of degree $m-1$, then z is always the m th derivative of an admissible variation, an admissible variation being one such that its m th derivative is integrable and such that it satisfies conditions (4.2). For by integrating z by parts m times from a to x , a function η is determined whose $(m-1)$ -st derivative is continuous on (a,b) and is such that $\eta(a) = \eta'(a) = \dots = \eta^{(m-1)}(a) = 0$. Then

$$\int_a^b g z \, dx = \int_a^b g \eta^{(m)} \, dx = g(b) \eta^{(m-1)}(b) + \dots \pm g^{(m-1)}(b) \eta(b) = 0$$

for an arbitrary polynomial g , i. e. for arbitrary values $g(b)$, $g'(b)$, $\dots$, $g^{(m-1)}(b)$. Hence $\eta(b) = \eta'(b) = \dots = \eta^{(m-1)}(b) = 0$ and $\eta(x)$ is an admissible variation. If N is

integrable in (a, b) and if $g(x) \equiv u_1 + u_2 x + \dots + u_m x^{m-1}$, then

$$U(u_1, \dots, u_m) \equiv \int_a^b (N - g)^2 dx \geq 0$$

has a minimum for some set of values $u_1 = \bar{u}_1, u_2 = \bar{u}_2,$

$$u_3 = \bar{u}_3, u_4 = \bar{u}_4, \dots, u_m = \bar{u}_m,$$

i. e. for the polynomial $\bar{g}(x) \equiv \bar{u}_1 + \bar{u}_2 x + \dots + \bar{u}_m x^{m-1}$.

Now if g is any arbitrary polynomial of degree $m-1$ and t is a parameter independent of x ,

$$U(t) \equiv \int_a^b (N - \bar{g} - t g)^2 dx \geq \bar{U} \equiv U(0).$$

Since $U(t)$ has a minimum at $t = 0$, its first derivative must vanish there,

$$(4.8) \quad -\frac{1}{2} U'(0) = \int_a^b (N - \bar{g}) g dx = 0$$

for arbitrary coefficients of $g(x)$. From the property arising from (4.7) stated and proved above, $N(x) - \bar{g}(x)$ is the m th derivative of an admissible variation. But

$$\int_a^b N n^{(m)} dx = 0,$$

and hence

$$\int_a^b N \lambda dx = \int_a^b (\bar{g} + \lambda) \lambda dx = 0,$$

where $\lambda \equiv N(x) - \bar{g}(x)$. With $g = \bar{g}$ in equation (4.8),

$$\int_a^b \lambda \bar{g} dx = 0.$$

Then

$$\int_a^b N \lambda \, dx = \int_a^b \lambda^2 \, dx = \int_a^b (N - \bar{g})^2 \, dx = 0$$

and $N = \bar{g}(x) = u_0 + u_1 x + \dots + u_m x^{m-1}$ for intervals on which $N(x)$ is continuous.

4. Methods of Haar, Hobson, Kryloff. Haar, 1922 (79), gives the following proof in which it is assumed that N is continuous and that the class of functions $n(x)$ is that for which the functions n have continuous m th derivatives and vanish with their first $m-1$ derivatives at $x = a$, $x = b$. For

$a \leq \alpha < \beta \leq b$ let

$$n(x) \equiv 0 \quad \text{on} \quad a \leq x \leq \alpha, \quad \beta \leq x \leq b,$$

$$n(x) \equiv (x - \alpha)^{m+1} (x - \beta)^{m+1} \quad \text{on} \quad \alpha \leq x \leq \beta$$

Then if

$$\omega(\alpha, \beta) \equiv \int_{\alpha}^{\beta} N(x) \left[\frac{d^m}{dx^m} (x - \alpha)^{m+1} (x - \beta)^{m+1} \right] dx = 0,$$

$$(4.10) \quad \frac{\partial^2 \omega}{\partial \alpha \partial \beta} = (m+1)^2 \int_{\alpha}^{\beta} N(x) \frac{d^m}{dx^m} [(x - \alpha)^m (x - \beta)^m] dx = 0.$$

Differentiation with respect to β gives

$$(m-1)! (\beta - \alpha)^m N(\beta) + (-1)^m \int_{\alpha}^{\beta} N \frac{d^m}{dx^m} [(x - \alpha)^m (x - \beta)^{m-1}] dx = 0.$$

Since the second term has a continuous derivative with respect to β , $N(x)$ has a continuous derivative on (a, b) except possibly at $x = \alpha$. Since α can be chosen arbitrarily, it

follows that $N(x)$ possesses a continuous derivative $N'(x)$ on the whole interval. From equation (4.10) by integration by parts

$$\int_{\alpha}^{\beta} N'(x) \frac{d^{m-1}}{dx^{m-1}} [(x-\alpha)^m (x-\beta)^m] dx = 0.$$

This is analogous to (4.10). Repetition of the previous argument shows the existence and continuity of $N''(x)$ satisfying

$$\int_{\alpha}^{\beta} N''(x) \frac{d^{m-2}}{dx^{m-2}} [(x-\alpha)^{m-1} (x-\beta)^{m-1}] dx = 0.$$

Further repetitions of the argument establish successively the existence and continuity of the 3rd, 4th, ..., m th derivatives of $N(x)$, the last satisfying

$$\int_{\alpha}^{\beta} N^{(m)}(x) (x-\alpha)(x-\beta) dx = 0$$

for every α, β . Hence $N^{(m)} \equiv 0$ and N is a polynomial of degree $m-1$. Haar himself says of his proof, "Es ist bemerkenswert an diesem verfahren, dass zuerst die Differenzierbarkeit der Funktion bewiesen wird, und daraus auf die Bauart der Funktion geschlossen wird."

Hobson, 1913 (72), makes use of his proof of the fundamental lemma to show that the conclusion of the Du Bois Reymond lemma, that N is a polynomial of degree $m-1$, holds almost everywhere on (a,b) if it is assumed only that N is summable and the class of functions η is that for which η are finite polynomials vanishing with their first $m-1$

differential coefficients at $x = a$, $x = b$.

Hobson, 1913 (81), also first proved the following theorem of which Du Bois Reymond's lemma is a special case.

Theorem. Let q be a continuous function of x of limited variation on (a, b) and let $q_1, \dots, q_p$ be functions of x having continuous differential coefficients of order 1, 2, ..., p respectively ^I throughout the interval. Let $f(x)$ be a function summable in (a, b) and such that

$$(4.11) \quad \int_a^b \left(q_0 \eta + q_1 \frac{d\eta}{dx} + \dots + q_p \frac{d^p \eta}{dx^p} \right) f dx = 0$$

for every value of $\eta(x)$ consisting of a finite polynomial which together with its first $p-1$ differential coefficients vanishes at $x = a$, $x = b$. Then if f be continuous on (a, b) , its first p differential coefficients all exist and are continuous interior to (a, b) and f satisfies the equation

$$(4.12) \quad \Lambda(f) \equiv \frac{d^p}{dx^p} (q_p f) - \frac{d^{p-1}}{dx^{p-1}} (q_{p-1} f) + \frac{d^{p-2}}{dx^{p-2}} (q_{p-2} f) - \dots + (-1)^p q_0 f = 0$$

which is the equation adjoint to the equation

$$q_0 \eta + q_1 \eta' + \dots + q_p \eta^{(p)} = 0.$$

I) Consequently $q_1, \dots, q_p$ are also of limited variation on (a, b) .

If $f(x)$ is continuous only in some interval contained in (a, b) , its value must satisfy the adjoint equation at all points in the interior of the interval of continuity.

For proof, Hobson integrates equation (4.1) by parts obtaining the equation

$$\int_a^b \left[q_p f - \int_a^x q_{p-1} f dx + \int_a^x dx \int_a^x q_{p-2} f dx - \dots \right] \frac{d^p \eta}{dx^p} dx = 0.$$

He then makes use of Du Bois Reymond's lemma of page 31, so although the theorem as stated contains Du Bois Reymond's lemma as a special case, the proof depends on that lemma.

Haar, 1922 (82), proves this theorem under slightly different hypotheses. He assumes that f is continuous on (a, b) , $q_p \equiv 1$, and $q_0, q_1, \dots, q_{p-1}$ are differentiable an indefinite number of times. If with these changes, equation (4.11) holds for all $\eta(x)$ having continuous p th derivatives and vanishing with their first $p-1$ derivatives at $x = a, x = b$, then f is differentiable p times and a solution of the adjoint equation (4.12).

The function $\eta(x)$ is defined as above (4.9) in Haar's proof of Du Bois Reymond's lemma. In a similar manner it is shown that f is differentiable p times. With η chosen as in (4.9) and integration by parts, equation (4.11) becomes

$$\int_a^b \Lambda(f)(x-a)^p(x-b)^p dx = 0.$$

Operated on p times by $\partial^2 / \partial \alpha \partial \beta$ this gives

$$\int_a^b \Lambda(f) dx = 0$$

for every α, β , and hence $\Lambda(f) = 0$.

With $f \equiv 1$, and with the assumption that equation (4.11) holds for the class of functions $\eta(x)$ for which $\eta(x)$ have continuous p th derivatives and vanish with their $p-1$ derivatives at $x = a$, $x = b$, Kryloff, 1923 (33) states the following relation between the q_i as a consequence of (4.11):

$$\frac{d}{dx} \left\{ \dots \left\{ \frac{d}{dx} \left\{ \frac{d}{dx} \left[\frac{dq_p}{dx} - q_{p-1} \right] + q_{p-2} \right. \right. \right. \right. \\ \left. \left. \left. - q_{p-3} \right\} \dots \right\} = (-1)^p q_0(x), \right.$$

where, however, the q_i are not necessarily differentiable, but only the combinations of them appearing above. Without proof Kryloff states that one is led to this conclusion by generalizing the method used by him for the fundamental lemma and Du Bois Reymond's lemma. In § 6 of this chapter, attention will be called to what seem to be inaccuracies in Kryloff's methods. Kryloff also states as being easily provable a result proved and used by Zermelo^I, that if (4.7)

I) See p. 58 of this paper.

holds, where Z is integrable and $g(x)$ a polynomial of degree $m-1$, then Z is the m th derivative of an admissible variation $\eta(x)$.

5. Hilbert's proof of Du Bois Reymond's lemma for $m = 1$ and its relation to a method used by Gauss. Whittemore, 1901 (84), presents Hilbert's proof for the case $m = 1$. It is assumed that y' has only a finite number of discontinuities, although Whittemore remarks that his method applies without modification if y' has an infinite number of discontinuities as long as the content of this set of points of discontinuity is zero.

Two intervals (α_1, β_1) , (α_2, β_2) are chosen so that

$$a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < b, \quad \beta_1 - \alpha_1 = \beta_2 - \alpha_2$$

and such that on $\alpha_1 \leq x \leq \beta_1$, $\alpha_2 \leq x \leq \beta_2$, $y'(x)$

is continuous. The function $\eta'(x)$ is chosen in this way:

$$\eta' = 0 \quad \text{on} \quad a \leq x \leq \alpha_1,$$

$$\eta' = \varepsilon(x - \alpha_1)(\beta_1 - x) \quad \text{on} \quad \alpha_1 < x \leq \beta_1,$$

$$\eta' = 0 \quad \text{on} \quad \beta_1 < x \leq \alpha_2$$

$$\eta' = -\varepsilon(x - \alpha_2)(\beta_2 - x) \quad \text{on} \quad \alpha_2 < x \leq \beta_2$$

$$\eta' = 0 \quad \text{on} \quad \beta_2 < x \leq b,$$

$\varepsilon =$ positive constant.

With the function $\eta'(x)$ thus defined substituted in equation

$$(4-13) \quad \int_a^b N \eta' dx = 0,$$

and the variable x changed to $x + \alpha_1 - \alpha_2$ on (α_2, β_2)

it follows that

$$\int_{\alpha_1}^{\beta_1} \varepsilon(x - \alpha_1)(\beta_1 - x) [N(x) - N(x + \alpha_2 - \alpha_1)] dx = 0.$$

Since $\varepsilon(x - \alpha_1)(\beta_1 - x)$ is never negative on (α_1, β_1) , the term in brackets must vanish for some value of x in (α_1, β_1) .

Hence at $x = \alpha_1 + \theta(\beta_1 - \alpha_1)$, $0 \leq \theta \leq 1$,

$$N(\alpha_1 + \theta(\beta_1 - \alpha_1)) = N(\alpha_2 + \theta(\beta_2 - \alpha_2)).$$

As $\beta_1 - \alpha_1$ approaches zero, $N(\alpha_1) = N(\alpha_2)$; but since α_1, α_2 were any two points in (a, b) at which y' is continuous, N is constant on (a, b) except perhaps at points of discontinuity of y' . Whittemore shows further that at a point of discontinuity, $x = c$, $N(c-0)$ and $N(c+0)$ exist and are equal to the constant value of N at points of continuity.

This is the proof given by Bolza, 1909 (86), and is also essentially that given by Gernet, 1902 (88).

Bolza, 1921? (87), has noted the striking similarity between this proof and one by Gauss, 1840 (85), for a problem in potential theory. This problem is concerned with the theorem that the density $m(\xi, \eta, \zeta)$ of a continuous distribution of matter at points (ξ, η, ζ) of a given surface S can always be so chosen, and in only one way, that

its potential at (x, y, z)

$$V(x, y, z) = \int m \frac{ds}{r}$$

will be equal to a given continuous function U of the coordinate of a point (ξ, η, ζ) of the surface, where r is the distance of the point (ξ, η, ζ) of the surface element ds from the given point (x, y, z) , and where the integral is taken over the whole surface. The calculus of variations problem involved is that of finding among all such continuous distributions m for which

$$m \geq 0, \quad \int m \, ds = M,$$

that for which

$$\Omega = \int (V - 2U) m \, ds$$

taken over the surface s is a minimum.

If the supposed minimizing function m is replaced by a neighboring function $m + u$, where u is continuous on the surface, it is shown that for $W \equiv V - U$,

$$\delta \Omega = 2 \int W u \, ds.$$

For a minimum, $\delta \Omega \geq 0$ for all u sufficiently small in absolute value satisfying

$$m + u \geq 0, \quad \int u \, ds = 0.$$

It follows that W is a constant. For if W were not constant, let A be some mean value of W . Then there would exist two equal portions p and q of the surface with the mass so distri-

buted over them that $W - A > 0$ in p , and $W - A < 0$ in q .

If $u = -v$ in p , $u = v$ in q , where $v > 0$ and if $u = 0$ at all other parts of the surface, then $\int \Omega < 0$ since $\int \Omega$ can be given the form.

$$\int \Omega = 2 \int (W - A) u \, d s.$$

Bolza notes that the function u that is used is not continuous on the surface and shows how it can be slightly modified by choosing for u a continuous function like Hilbert's.

6. Proofs of Du Bois Reymond's lemma for the case $m = 1$ by Hahn, Jordan, Tonelli, Jacobsthal, Goursat, Bliss, Kurosu and Kryloff. Hahn, 1907 (89), considers the plane problem in parametric form. It is assumed that N is Lebesgue integrable and that $\eta(x)$ has a continuous derivative except perhaps at a finite number of points on (a, b) . The function used for $\eta(x)$ in the proof is defined by the following conditions:

$$\begin{array}{ll} \eta(x) = 0 & \text{on } (a, \sigma_0), \\ \eta(x) = x - \sigma_0 & \text{on } (\sigma_0, \sigma_0 + \sigma), \\ \eta(x) = \sigma & \text{on } (\sigma_0 + \sigma, \sigma_1), \\ \eta(x) = \sigma - (x - \sigma_1) & \text{on } (\sigma_1, \sigma_1 + \sigma), \\ \eta(x) = 0 & \text{on } (\sigma_1 + \sigma, b). \end{array}$$

Using the same type of function for $\eta(x)$, Jordan, 1915 (90), and Tonelli, 1923 (91), give proofs similar to Hahn's.

Jacobsthal, 1910 (92), has the following simple proof. He assumes that $N(x)$ is continuous on (a, b) and that (4-13) holds for all $\eta(x)$ vanishing at $x = a$, $x = b$ and having continuous first derivatives. An $\eta(x)$ satisfying these conditions is

$$(4-14) \quad \eta(x) = \int_a^x \left(N + \frac{\int_a^b N dx}{a-b} \right) dx.$$

For

$$B \equiv \frac{\int_a^b N dx}{a-b}$$

it follows from (4-13) that

$$\int_a^b (N + B) \eta' dx = 0,$$

which with the η' of (4-14) gives

$$\int_a^b (N + B)^2 dx = 0.$$

Then $N = -B =$ a constant.

This is also the proof given by Goursat, 1927 (93), and by Bliss, 1925 (94). The proof holds if N and η' have a finite number of discontinuities and the lemma is proved under these conditions in the latter reference.

Kurosu, 1915 (95), gives an indirect proof where it is assumed that $N(x)$ and $\eta'(x)$ are continuous on (a, b) . If N is not constant on (a, b) there exist two points

ξ_1, ξ_2 of the interval such that $N(\xi_1) \neq N(\xi_2)$. The function used for $\eta(x)$ is

$$\eta(x) = e^{\frac{\rho}{(x-\xi_1)(x-\xi_2)}} \quad \text{on} \quad \xi_1 < x < \xi_2, \quad \rho > 0,$$

$$\eta(x) = 0 \quad \text{on} \quad a \leq x \leq \xi_1, \quad \xi_2 \leq x \leq b.$$

For ρ sufficiently small this function leads to a contradiction for it is shown that it gives to the integral in (4.13) a value not zero.

Haar, 1922 (96), gives a short proof under the same hypothesis. For $\eta(x)$ he has

$$\eta(x) = 0 \quad \text{on} \quad a \leq x \leq \alpha, \quad \beta \leq x \leq b, \quad \alpha < \beta,$$

$$\eta(x) = (x-\alpha)^2(x-\beta)^2 \quad \text{on} \quad \alpha \leq x \leq \beta,$$

where α, β are arbitrary points of (a, b) . He defines

$$\omega(\alpha, \beta) = \int_{\alpha}^{\beta} N(x) \frac{d}{dx} [(x-\alpha)^2(x-\beta)^2] dx = 0.$$

$$\text{Then} \quad \partial^4 \omega / \partial \alpha^2 \partial \beta^2 = 4 [N(\beta) - N(\alpha)] = 0 \quad \text{and}$$

hence $N(\alpha) = N(\beta)$, but since α, β were arbitrary, N = constant.

Kryloff, 1923 (97), seems to argue that when

$$(4.15) \quad \int_a^b N \eta' dx = 0$$

for all $\eta(x)$ such that $\eta(a) = \eta(b) = 0$, it must also be satisfied for all functions $\eta'(x)$ whatsoever satisfying

$\eta'(a) = \eta'(b) = 0$ so that the fundamental lemma of page 3 would give $N = 0$. But this is evidently incorrect, for equation (4.15) holds for all $\eta(x)$ such that $\eta(a) = \eta(b) = 0$, and not every function $\eta'(x)$ satisfying $\eta'(a) = \eta'(b) = 0$ is the derivative of a function $\eta(x)$ such that $\eta(a) = \eta(b) = 0$. Furthermore, from remarks made in § 2 of this chapter, or from Hilbert's proof of Du Bois Reymond's lemma for $m = 1$, it is seen that N must be a constant if equation (4.15) holds for $\eta(x)$ for which $\eta(a) = \eta(b) = \eta'(a) = \eta'(b) = 0$.

7. Theorems of Bliss and Razmadze. Bliss, 1908 (98), derives the Euler-Lagrange equation in the Du Bois Reymond form for the case $m = 1$ without using either the Lagrange or Du Bois Reymond integration by parts. This is done by means of the following

Theorem . . . If

$$I \equiv \int_a^b [A(x, y(x)) + B(x, y(x))y'(x)] dx,$$

where A, B are continuous in a certain region R of xy space, and if $y = y(x)$ joins two given points $(a, y_1), (b, y_2)$, lies in R , and has a continuous first derivative, then a necessary and sufficient condition for I to be independent of the path in R is that there exist a single valued function $\phi(x, y)$ in R having derivatives $\partial\phi/\partial x = A, \partial\phi/\partial y = B$.

Later Razmadze, 1921 (99), proves this

Theorem. Let $M(x), N(x)$ be two continuous functions on (a, b) . If

$$I \equiv \int_a^b [M(x) y(x) + N(x) y'(x)] dx = 0$$

for all possible functions $y(x)$, having a continuous first derivative and vanishing at $x = a$, $x = b$, then dN/dx exists and is continuous on (a, b) and $dN/dx = M$.

It is clear that this is a special case of the theorem which Bliss uses. For $A = M(x) y(x)$, $B = N(x)$, $y_1 = y_2 = 0$ the theorem used by Bliss gives the existence of $\phi(x, y)$ such that $\partial\phi/\partial x = My$, $\partial\phi/\partial y = N$. But $\partial^2\phi/\partial x\partial y = M$ exists and is continuous, and N is continuous, and hence $\partial^2\phi/\partial y\partial x = dN/dx$ exists and equals M , which is the result obtained by Razmadze.

This theorem contains the fundamental lemma for $N \equiv 0$ and Du Bois Reymond's lemma for $M \equiv 0$ as special cases. The theorem concerning independence of the path assumes that y has continuous first derivatives. The Euler - Lagrange equation, however, is derived by Bliss under the assumption that y is made up of a finite number of such arcs.

7. A generalization by the author of Jackson's proof of Du Bois Reymond's lemma. The following proof is a generalization of a proof given by Jackson, 1922 (100), of Du Bois Reymond's lemma for the case $m = 2$.

It is assumed that $N(x)$ is continuous on $a \leq x \leq b$ and the class of functions η is that for which η has continuous m th derivative on (a, b) and for which conditions (4.2) are satisfied. Then the determinant of the following matrix vanishes for all continuous $f_1(x), f_2(x), \dots, f_{m+1}(x)$.

$$\Delta \equiv \begin{vmatrix} \int_a^b N \zeta_1 dx & \int_a^b N \zeta_2 dx & \cdots & \int_a^b N \zeta_{m+1} dx \\ \int_a^b x^{m-1} \zeta_1 dx & \int_a^b x^{m-1} \zeta_2 dx & \cdots & \int_a^b x^{m-1} \zeta_{m+1} dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b x \zeta_1 dx & \int_a^b x \zeta_2 dx & \cdots & \int_a^b x \zeta_{m+1} dx \\ \int_a^b \zeta_1 dx & \int_a^b \zeta_2 dx & \cdots & \int_a^b \zeta_{m+1} dx \end{vmatrix}$$

For suppose there exist continuous $\zeta_1, \zeta_2, \dots, \zeta_{m+1}$ such that $|\Delta| \neq 0$. Then there exist constants $C_1, \dots, C_{m+1}$ not all zero such that

$$C_1 \int_a^b \zeta_1 dx + C_2 \int_a^b \zeta_2 dx + \cdots + C_{m+1} \int_a^b \zeta_{m+1} dx = 0$$

$$C_1 \int_a^b x \zeta_1 dx + C_2 \int_a^b x \zeta_2 dx + \cdots + C_{m+1} \int_a^b x \zeta_{m+1} dx = 0$$

(4.16) $\dots \dots \dots$

$$C_1 \int_a^b x^{m-1} \zeta_1 dx + C_2 \int_a^b x^{m-1} \zeta_2 dx + \cdots + C_{m+1} \int_a^b x^{m-1} \zeta_{m+1} dx = 0$$

$$C_1 \int_a^b N \zeta_1 dx + C_2 \int_a^b N \zeta_2 dx + \cdots + C_{m+1} \int_a^b N \zeta_{m+1} dx \neq 0.$$

Let

$$(4.17) \quad \eta(x) = \int_a^x \int_a^x \cdots \int_a^x (C_1 \zeta_1 + \cdots + C_{m+1} \zeta_{m+1}) dx^m.$$

Obviously $\eta, \eta', \dots, \eta^{(m-1)}$ vanish at $x = a$. By integrating (4.17) by parts it is seen that $\eta(1) = 0$, and it can also be shown similarly that $\eta'(b) = \eta''(b) = \dots = \eta^{(m-1)}(b) = 0$. Since (4.17) defines an admissible η , then by the hypothesis of the lemma

$$\int_a^b N(c_1 y_1 + \dots + c_{m+1} y_{m+1}) dx = 0$$

which contradicts the last of equations (4.15). Hence $|\Delta| = 0$ for all continuous $f_1, f_2, \dots, f_{m+1}$.

Then $\gamma_1, \dots, \gamma_m$ can be selected so that the matrix Δ is of such rank m and such that the determinant of the matrix obtained from Δ by suppressing the first row and last column does not vanish. The constants $c_1, c_2, \dots, c_m$ can be selected so that

$$\int_a^b N \zeta_1 dx - C_1 \int_a^b x^{m-1} \zeta_1 dx - \dots - C_m \int_a^b \zeta_1 dx = 0$$

$$\dots$$

$$\int_a^b N \zeta_m dx - C_1 \int_a^b x^{m-1} \zeta_m dx - \dots - C_m \int_a^b \zeta_m dx = 0.$$

Since $|\Delta| = 0$ for every set of continuous functions,

$S_1, S_2, \dots, S_{m+1}$, it follows that

$$\int_a^b (N - C_1 x^{m-1} - \dots - C_m) \zeta(x) dx = 0$$

for every continuous $\gamma(x)$. Hence

$$N = C_1 x^{m-1} + C_2 x^{m-2} + \dots + C_m.$$

It will be noted that the function $\eta(x)$ defined by (4.17) is identical with that chosen by Jacobsthal.

CHAPTER V

ANALOGUES OF THE FUNDAMENTAL LEMMA AND DU BOIS REYMOND'S LEMMA FOR MULTIPLE INTEGRALS

1. Introduction. For the problem of minimizing the integral

$$(5.1) \quad J = \iint_B f(x, y, z, p, q) \, dx \, dy,$$

$$p = \partial z / \partial x, \quad q = \partial z / \partial y,$$

in a class of surfaces $z = Z(x, y)$ bounded by a closed space curve L , where the integration is to be taken over the projection B of the surface on the xy plane, the vanishing of the first variation is expressed by the equation,

$$(5.2) \quad \iint_B (f_x \delta + f_y \delta_y + f_q \delta_q) \, dx \, dy = 0$$

This is to hold for all $\delta(x, y)$ vanishing on the boundary of B . Three methods of the transformation of the first variation have been used to obtain conditions that $z(x, y)$ must satisfy as a consequence of (5.2). With these three methods are associated the names of Lagrange, Hilbert, and Haur.

If it is assumed that the second partial derivatives of z exist and are continuous, the transformation of (5.2) by the Lagrange integration by parts leads to the investiga-

tion of the following analogue of the fundamental lemmas.

Lemma A. If $M(x,y)$ is continuous in the closed region B and if

$$(5.3) \quad \iint_B M(x,y) \, \zeta(x,y) \, dx \, dy = 0$$

for all $\zeta(x,y)$ which have continuous first partial derivatives, and vanish on the boundary of B , then $M \equiv 0$.

When partial derivatives of higher order occur in the integrand of the integral to be minimized, the vanishing of the first variation can also be expressed in the form (5.3), where now, however, higher derivatives of ζ vanish on the boundary of B .

Lagrange seems to be the first to have formulated a calculus of variations problem involving multiple integrals. With B a rectangular region with sides parallel to the coordinate axes, $M = 0$ seemed a self evident consequence of (5.3) to many of the early writers. The proofs given by writers before Sarrus are open to the same criticisms as their proofs of the lemma for simple integrals. Sarrus' proof, 1848 (114), though valid, assumes the existence of partial derivatives of the third order of z for the problem of double integrals of the type (5.1), for which first partial derivatives of z occur in the integrand of the integral to be minimized. Bolza, 1909 (119), gives a proof for the case when z has continuous second partial derivatives. This proof and one by Kneser, 1925 (121) can be extended at once to multiple integrals of every order.

Hilbert, 1904 (122), proves a lemma which answers the du Bois Reymond objection for a double integral calculus of variations problem where third partial derivatives of z appear in the integrand. Mason, 1905 (123), has generalized the lemma for the case when partial derivatives of any order occur in the integrand. The regions of integration are assumed to be rectangles R with sides parallel to the coordinate axes. By what will hereafter be called the Hilbert integration by parts, the vanishing of the first variation of a double integral involving partial derivatives of any order can be expressed by the equation

$$(5.4) \quad \iint_R F(x, y) \frac{\partial^{m+n} \zeta(x, y)}{\partial x^m \partial y^n} dx dy = 0,$$

which holds for all admissible $\zeta(x, y)$. Admissible variations $\zeta(x, y)$ are such that $\partial^{i+j} \zeta / \partial x^i \partial y^j$ ($i = 0, \dots, m$; $j = 0, \dots, n$) exist and are continuous on $a \leq x \leq \alpha$, $b \leq y \leq \beta$, where (a, b) , (a, β) , (α, b) , (α, β) are the vertices of R ; $\zeta(x, y)$, $\partial^i \zeta / \partial x^i$, ($i = 1, \dots, m-1$) vanish at $x = a$, $x = \alpha$, identically in y ;

$$\zeta(x, y), \partial^j \zeta / \partial y^j, \quad (j = 1, \dots, n-1)$$

vanish at $y = b$, $y = \beta$ identically in x .

Lemma B. If equation (5.4) holds for all admissible

$\zeta(x, y)$, then F has the form

$$(5.5) \quad F = \sum_{i=0}^{m-1} x^i Y_i + \sum_{j=0}^{n-1} y^j X_j,$$

where Y_i , X_j are continuous functions of y and x respectively.

For $m = n = 3$ this is the lemma proved by Hilbert.

Mason proved it for the case $m = n$, and Kubota, 1916 (126), gave a proof later with m and n not necessarily equal.

A third method of transformation of the first variation has been given by Haar, 1919 (128), who expresses (5.2) in the form

$$(5.6) \quad \iint_B (u \zeta_x + v \zeta_y) dx dy = 0.$$

Haar then proves the following theorem which answers the Du Bois Reymond objection for double integrals involving only first partial derivatives.

Theorem. If u, v are continuous in a region B and such that (5.6) holds for all $\zeta(x, y)$ that vanish on the boundary of B , and have continuous first partial derivatives in the interior of B , then

$$(5.7) \quad \int_C u dy - v dx = 0$$

for every closed curve C in the interior of B .

Haar also proves this theorem when u and v are only assumed to be bounded and measurable, and $\zeta(x, y)$ satisfies a Lipschitz condition. Lichtenstein's proof, 1924 (129), of the Haar theorem seems to contain some inaccuracies which are corrected by the author of this paper in §5 below.

Another method of obtaining the Haar theorem is given by Schauder, 1928 (131), who makes use of Legendre polynomials. In the course of his proof he obtains a result which would follow from the Mason lemma for the case $m = 1$, a fact which Schauder apparently did not recognize.

Of the two methods of answering the Du Bois Reymond objection for double integrals, that of Mason is the most general; for the Mason lemma applies to a calculus of variations problem where partial derivatives of any order occur in the integrand of the integral to be minimized, while Haar's theorem applies when first derivatives only occur. Furthermore, as becomes evident through Schauder's proof, the Haar theorem can be obtained as a consequence of the Mason lemma for the case $m = 1$.

The analogue for triple integrals of the Haar theorem has been given by Szücs, 1927 (130). It is a direct extension of Lichtenstein's method and, as Szücs remarks, could be extended to prove the analogous theorem for any multiple integral.

In § 7 Mason's lemma is extended by the author of this paper to triple integrals and the analogous lemma for a multiple integral of any order is suggested. It is shown that Mason's method can even be used to obtain Du Bois Reymond's lemma for simple integrals. By means of three auxiliary lemmas the author has extended Kubota's method to triple integrals. Further by the use of Legendre polynomials, the analogue of Mason's lemma with $m = 1$ for triple integrals is obtained. In the concluding section it is shown how the first variation of a double integral can be transformed to give expressions to which the analogue of the fundamental lemma, Mason's lemma and Haar's theorem can be applied. Analogous transformations for triple integrals are also indicated.

2. Analogue of the fundamental lemma for multiple

integrals. A problem of the calculus of variations involving multiple integrals seems to have been first formulated by Lagrange, (101). He considers the problem of minimizing

$$\iint \sqrt{1 + p^2 + q^2} \, dx \, dy$$

For this integral M in (5.3) is given by

$$M \equiv - \frac{\partial}{\partial x} \frac{p}{\sqrt{1 + p^2 + q^2}} - \frac{\partial}{\partial y} \frac{q}{\sqrt{1 + p^2 + q^2}}$$

Lagrange concludes from the arbitrary character of $f(x, y)$ that $M = 0$.

Other writers without stating continuity properties that must be satisfied in order that the vanishing of the first variation of a double integral can be expressed in the form (5.3), and without proof, conclude that $M = 0$ necessarily follows from (5.3). Among these are Brunacci, 1808 (102), Poisson, 1833 (103), Abbott, 1836 (104), Jacobi, 1837 (105), Cournot, 1846 (106), Delannay, 1842 (107), Navier, 1875 (108), Carll, 1881 (109), Picart, 1883 (110), and Sabinine, 1899 (111).

Proofs that $M = 0$ is a consequence of (5.3) and proofs of the analogous lemma for multiple integrals of any order are obtained by extending the proofs given for simple integrals. The methods of Dirksen, 1823 (112), and Ohm, 1825 (113), are open to the same criticisms as their methods for simple integrals. In their proofs B is a rectangle R with sides parallel to the coordinate axes. With $M = 0$ deduced for every rectangle interior to a simply connected region B , it follows that

$M \equiv 0$ in B . However, this is not stated explicitly by the early writers. Sarrus' proof, 1848 (114), has already been mentioned on page 24 of this paper. His proof assumes for the problem of double integrals for which first partial derivatives of z occur in the integrand, the existence and continuity of third partial derivatives of z . Moigno - Lindelöf, 1861 (115), give Sarrus' proof. Stegmann, 1854 (116), and Strauch, 1854 (117), refer to their proofs for simple integrals.

Kobb, 1892 (118), makes an argument similar to Cauchy's for simple integrals. M is assumed to be continuous. Kobb supposes that $f(x, y) = 0$ is the equation of the boundary curve of B and sets $\int(x, y) = f(x, y) \psi_1(x, y)$. The function $\psi_1(x, y)$, chosen to have the same sign as Mf , gives the integral a positive value unless $M = 0$ interior to B . The vanishing of M on the boundary follows from the continuity of M .

Bolza, 1909 (119), gives a proof for the case when (5.3) holds for all $\int(x, y)$ which have continuous first partial derivatives and vanish on the boundary of B , and where it is assumed that M is continuous in the closed region B . If $M \neq 0$, e.g. $M > 0$, at some interior point P_0 of B , ρ can be chosen so small that $M > 0$ on the circular surface C with radius ρ , center P_0 , and such that at the same time this circle lies interior to B . Then for

$$\int = \left[\rho^2 - (x - x_0)^2 - (y - y_0)^2 \right]^2 \quad \text{in } C,$$

$$\int = 0 \quad \text{outside } C,$$

the integral becomes positive contrary to hypothesis and hence $M = 0$ in the interior of B and, from the continuity of M , on the boundary of B . Goursat, 1927 (120), also gives this proof.

Kneser's proof, 1925 (121), is the analogue of that of Weierstrass for simple integrals. If $M \neq 0$, e. g. $M > 0$, at some point P_0 in B , then for some rectangle, $x_0 \leq x \leq x_1$, $y_0 \leq y \leq y_1$, M is also positive. For

$$J(x, y) = (x - x_0)^{k+1} (x_1 - x)^{k+1} (y - y_0)^{k+1} (y_1 - y)^{k+1}$$

for x in (x_0, x_1) , y in (y_0, y_1) , and zero elsewhere, the integral in (5.3) remains positive. Hence $M = 0$.

Either of these proofs can be extended at once to multiple integrals of every order.

3. The methods of Hilbert, Mason, Fubini, Kurosu, Kubota. Hilbert, 1904 (122), proves Lemma B of page 56 with $m = n = 3$. Mason's method, 1905 (123), of proving this lemma for every m with $m = n$, is very general in its application. For this reason his proof will be given here. It will be noted that the conditions imposed on $J(x, y)$ in the definition of admissible variations $J(x, y)$, given on page 56, imply that $\partial^{k+p} J / \partial x^k \partial y^p = 0$, ($k = 0, \dots, m-1$; $p = 0, \dots, m$) for $x = a, x = \alpha$ identically in y ;
 $\partial^{p+k} J / \partial x^p \partial y^k = 0$, ($k = 0, \dots, m-1$; $p = 0, \dots, m$) for $y = b, y = \beta$ identically in x . By using these relations, integration

by parts shows that (5.5) is sufficient for (5.4) without further restrictions on $F(x, y)$. Then

$$(5.8) \quad \int_b^{\beta} \int_a^{\alpha} \left[F(x, y) - \sum_{k=0}^{m-1} x^k Y_k + y^k X_k \right] \frac{\partial^{2m} \mathfrak{F}}{\partial x^m \partial y^m} dx dy = 0$$

holds for all admissible $\mathfrak{F}(x, y)$ and all continuous

$X_k(x), Y_k(y)$. $f_{00}, f_{10}, f_{11}, \dots, f_{mm}$

are defined by the following equations,

$$f_{00}(x, y) \equiv \int_{b(y)}^y \int_{a(x)}^x F(x, y) dx^m dy^m,$$

$$f_{10}(x, y) \equiv f_{00}(x, y) - \frac{(x-a)^m}{(\alpha-a)^m} f_{00}(\alpha, y),$$

$$f_{11}(x, y) \equiv f_{10}(x, y) - \frac{(y-b)^m}{(\beta-b)^m} f_{10}(x, \beta),$$

.....

$$f_{m-m-1}(x, y) \equiv f_{m-1, m-1}(x, y) - \frac{(x-a)^m (x-\alpha)^{m-1}}{(m-1)! (\alpha-a)^m} \left[\frac{\partial^{m-1} f_{m-1, m-1}(x, y)}{\partial x^{m-1}} \right]_{x=\alpha},$$

$$f_{mm}(x, y) \equiv f_{m, m-1}(x, y) - \frac{(y-b)^m (y-\beta)^{m-1}}{(m-1)! (\beta-b)^m} \left[\frac{\partial^{m-1} f_{m, m-1}(x, y)}{\partial y^{m-1}} \right]_{y=\beta}$$

then $\int (x, y) \equiv f_{mm} (x, y)$ is an admissible variation.

It is seen that

$$(5.9) \quad \frac{\partial^{2m} f(x, y)}{\partial x^m \partial y^m} = F(x, y) - \sum_{k=0}^{m-1} (x^k Y_k + y^k X_k),$$

where the X_k, Y_k are continuous functions of x, y respectively depending on $F(x, y)$. With (5.9) substituted in (5.8), it follows that

$$\int_a^\alpha \int_b^\beta \left[F - \sum_{k=0}^{m-1} (x^k Y_k + y^k X_k) \right]^2 dx dy = 0,$$

which gives the desired result

$$F = \sum_{k=0}^{m-1} (x^k Y_k(y) + y^k X_k(x)).$$

Fubini, 1906 (124), uses Hilbert's method to prove the lemma for $m = 2$. Kurosu, 1915 (125), obtains the conclusion of the lemma for $m = 1$ without proving it first for the case when B is a rectangle. The proof is made by defining $\int (x, y)$ as an exponential function interior to some rectangle R lying in B and identically zero in the rest of B not lying in R .

Another method has been used by Kubota, 1916 (126), to prove lemma B stated on page 82. This is accomplished with the aid of four auxiliary lemmas.

Lemma 1. If $N(x)$ is continuous on $a \leq x \leq \alpha$

and if

$$\int_a^\alpha N(x) \eta(x) dx = 0$$

for every continuous function $\eta(x)$ such that

$$\int_a^\alpha N_1 \eta \, dx = \int_a^\alpha N_2 \eta \, dx = \dots = \int_a^\alpha N_p \eta \, dx = 0,$$

where $N_1(x)$, $N_2(x)$, ..., $N_p(x)$ are continuous, then $N(x)$ is of the form

$$N(x) = c, N_1 + \dots + c_p N_p$$

where the c 's are constants.

Lemma 2. If $N(x, y)$ is continuous for $a \leq x \leq \alpha$, $b \leq y \leq \beta$ and if

$$\int_a^\alpha N(x, y) F(x) \, dx$$

has the form $B_1 \phi_1(y) + B_2 \phi_2(y) + \dots + B_m \phi_m(y)$, where $B_1, \dots, B_m$ are constants, for every continuous function $F(x)$, then $N(x, y)$ necessarily has the form

$$N(x, y) = b_1(x) \phi_1(y) + \dots + b_m(x) \phi_m(y).$$

Lemma 3. If

$$\int_a^\alpha N(x, y) F(x) \, dx$$

has the form $B_1 \phi_1(y) + \dots + B_m \phi_m(y)$ for every continuous function $F(x)$ satisfying

$$\int_a^\alpha f_i(x) F(x) \, dx = 0 \quad (i = 1, \dots, n),$$

then

$$N = b_1 \phi_1(y) + \dots + b_m(x) \phi_m(y) + a_1(y) f_1(x) + \dots + a_n(y) f_n(x).$$

Lemma 4. If

$$\int_a^\alpha \int_b^\beta N(x,y) \zeta(x,y) dx dy = 0$$

for every continuous $\zeta(x,y)$ satisfying

$$\int_a^\alpha \zeta(x,y) f_i(x) dx = 0 \quad (i = 1, \dots, n)$$

$$\int_b^\beta \zeta(x,y) \phi_j(y) dy = 0 \quad (j = 1, \dots, m)$$

then $N(x,y)$ necessarily has the form

$$N(x,y) = \sum_{j=1}^m b_j(x) \phi_j(y) + \sum_{i=1}^n a_i(y) f_i(x).$$

In order to prove Lemma B (p. 82) Kubota sets

$$\mathfrak{F}(x,y) \equiv \int_a^x \dots \int_a^x \int_b^y \dots \int_b^y \zeta(x,y) dx^m dy^m$$

where $\zeta(x,y)$ is an arbitrary continuous function. Obviously the conditions imposed on admissible functions $\mathfrak{F}(x,y)$ at $x = a$, $y = b$ (p.56) are satisfied. The conditions to be satisfied at $x = \alpha$; $y = \beta$ can be expressed by

$$\int_a^\alpha \mathfrak{F}(x,y) dx = 0, \int_a^\alpha x \mathfrak{F}(x,y) dx = 0, \dots, \int_a^\alpha x^{m-1} \mathfrak{F}(x,y) dx = 0,$$

$$\int_b^\beta \mathfrak{F}(x,y) dy = 0, \int_b^\beta y \mathfrak{F}(x,y) dy = 0, \dots, \int_b^\beta y^{n-1} \mathfrak{F}(x,y) dy = 0.$$

The desired lemma then follows from Lemma 4.

4. The Theorem of Haar. The conclusion (5.7) of Haar's Theorem given on page 57 is equivalent to saying that there exists a function $w(x,y)$ having continuous first partial derivatives such that

$$(5.10) \quad \partial w / \partial x = -v(x,y), \quad \partial w / \partial y = u(x,y).$$

Haar, 1919 (127) makes the proof in four steps.

It is first shown that if (5.6) holds for all $\gamma(x,y)$ vanishing on the boundary of B and having continuous first partial derivatives in B , that it also holds for such $\gamma(x,y)$, whose derivatives, however, may have finite jumps along a regular curve in B . The second step shows that (5.7) holds when C is any square lying in B with sides parallel to the coordinate axes. To prove this a special coordinate system is introduced: if x, y are the usual Cartesian coordinates, with origin O and $P(x,y)$ an arbitrary point, then one of the new coordinates is the angle that OP makes with the positive x axis; the other coordinate ρ , always positive, is chosen as the half side of the square through P , described with O as center, sides parallel to the axes. The function γ is chosen as a function of ρ alone, being a function defining a surface that has an edge in B , but for which (5.6) holds nevertheless by the first part of the proof. With (5.7) thus proved for all squares with sides parallel to the axes, the third step shows that (5.7) holds for all rectangles with

sides parallel to the axes. It follows then that the integral in (5.7) is independent of the path made up of broken line segments parallel to the axes. The final result is obtained by defining

$$w(x,y) \equiv \int_0^{P(x,y)} u \, dy - v \, dx$$

where OP is a path of the kind described. Now

$$w(x+h, y) - w(x, y) = \int_{x,y}^{x+h,y} u \, dy - v \, dx,$$

the path of integration being the straight line joining $(x+h, y)$ to (x, y) . It follows that $\partial w / \partial x = -v(x, y)$, and similarly, $\partial w / \partial y = u(x, y)$.

No restrictions on B are stated although from the method of proof it is seen that B must be simply connected.

Haar, 1927 (128), proves the theorem again when u and v are only assumed to be bounded and measurable and the variation $\int (x, y)$ satisfies a Lipschitz condition. It is first proved that (5.7) holds for almost all rectangles C interior to B with sides parallel to the coordinate axes, by a method very much like the earlier proof. However, the result is proved for rectangles at once by introducing a system of coordinates where one new coordinate of the point $P(x, y)$ referred to Cartesian coordinates with the origin O , is the angle made by OP with the positive x axis, and the

other coordinate is the half diagonal of the rectangle through P, center O, sides parallel to the Cartesian axes, and similar to an arbitrarily chosen rectangle. It is finally shown that (5.10) hold almost everywhere by applying twice a theorem of Fubini on measurable sets in higher dimensions.

5. A modification of Lichtenstein's method of proving Haar's theorem by the author of this paper. Szücs' proof of the analogous theorem for triple integrals. A simpler proof than Haar's can be made by remedying what seem to be incorrect details of a proof by Lichtenstein, 1924 (129). The functions $u(x,y)$, $v(x,y)$ are assumed continuous in and on the boundary of B, and (5.6) holds for all $\zeta(x,y)$ vanishing on the boundary of B and having continuous first partial derivatives.

Lichtenstein chooses a simply closed curve, Γ , lying in B, having no double points and having continuous curvature, Γ_h , a curve parallel to Γ , i. e. a curve every point of which is at a distance h along the outward normal of Γ from the corresponding point of Γ . The arc length s of Γ and the distance h along the normal are introduced as new independent variables. If Γ is represented by

$$\Gamma: \quad x = \xi(s), \quad y = \eta(s),$$

the transformation of coordinates is given by the equations

$$x = \xi(s) + h \, d\eta/ds$$

(5.11)

$$y = \eta(s) - h \, d\xi/ds$$

where $d\xi/ds = -\sin\alpha$, $d\eta/ds = \cos\alpha$, the angle α being the angle made by the normal with the positive x axis. Lichtenstein uses a function $\chi(h)$ with the following properties:

$$\chi(0) = 1, \chi(\varepsilon) = 0, \chi'(0) = 0, \chi'(\varepsilon) = 0, \quad (5.12)$$

$$\chi'(h) < 0, \quad 0 < h < \varepsilon.$$

Here a change will be made in Lichtenstein's proof, and $\chi(h)$ will be supposed to have properties (5.12) except that $\chi(0) = 0$. Then $\zeta(x, y)$ will be defined to be identically zero inside T and outside T_1 , while in the region between T and T_1 it will be given by $\zeta(x, y) = \chi(h)$. (Lichtenstein defines $\zeta(x, y)$ to be unity inside the curve T .) Now with the transformation (5.11), equation (5.6) becomes

$$(5.13) \quad \iint_{T_0}^{\varepsilon} \left\{ u \left(h \frac{d^2 \xi}{ds^2} - \frac{d\eta}{ds} \right) + v \left(h \frac{d^2 \eta}{ds^2} + \frac{d\xi}{ds} \right) \right\} \chi'(h) ds dh = 0.$$

For

$$w(h) \equiv \int_T \left\{ u \left(h \frac{d^2 \xi}{ds^2} - \frac{d\eta}{ds} \right) + v \left(h \frac{d^2 \eta}{ds^2} + \frac{d\xi}{ds} \right) \right\} ds,$$

equation (5.13) becomes

$$(5.14) \quad \int_0^{\varepsilon} w(h) \chi'(h) dh = 0.$$

By Du Bois Reymond's lemma, $w(h) = c$ where c is a constant.

This constant is zero¹. For $\chi(h)$ and $\xi(x,y)$ as defined by Lichtenstein are admissible and (5.14) therefore holds also for all $\chi(h)$ such that $\chi(0) = 1, \chi(\epsilon) = 0$. Since $w(h) = c$,

$$0 = \int_0^\epsilon c \chi'(h) dh = c \{ \chi(\epsilon) - \chi(0) \} = -c.$$

Hence $c = 0$ and

$$(5.15) \quad \int_{\Gamma} \left\{ u \left(h \frac{d^2 \xi}{ds^2} - \frac{d\eta}{ds} \right) + v \left(h \frac{d^2 \eta}{ds^2} + \frac{d\xi}{ds} \right) \right\} ds = 0.$$

For fixed h , equation (5.11) define a curve $\Gamma^{(h)}$ parallel to Γ at a distance h . Hence for $h = 0$ equation (5.15) can be written

$$(5.16) \quad \int_{\Gamma} v dx - u dy = 0.$$

Since Γ was arbitrary this gives the desired result for curves with continuous curvature. By considering curves with discontinuous curvature, for example rectangles, as limiting cases of curves with continuous curvature, the same result (5.7) would follow from the continuity of $u(x,y)$ and $v(x,y)$.

Szücs, 1927 (130), proves the following analogue of Haar's theorem.

1) Method to show $c = 0$ suggested by Prof. Bliss

Theorem. If $u(x, y, z)$, $v(x, y, z)$,

$t(x, y, z)$ are continuous functions in a three dimensional
region T and if

$$(5.17) \quad \iiint_T (u \xi_x + v \xi_y + t \xi_z) dx dy dz = 0$$

for all continuous $\xi(x, y, z)$ vanishing or reducing to a
constant on the bounding surface of T, having continuous first
partial derivatives in T, then

$$(5.18) \quad \iint_S (u \alpha + v \beta + t \gamma) dS = 0$$

for every closed surface S interior to T, without double
points, and having continuous tangent plane, where α, β, γ
are the direction cosines of the exterior normal to S.

The method of proof is a direct extension of Lichtenstein's method. Corresponding to the parallel curves T and T_1 , parallel surfaces S and S_1 are introduced. The distance along the normal to the surface S is chosen for one new coordinate, the other two being parameters which would locate a point on the surface.

Relations (5.10) obtained from Haar's theorem can be thought of as a parametric representation, with w as parameter, of all functions $u(x, y)$, $v(x, y)$ such that (5.7) vanishes for every closed curve in B . To obtain the analogue of relations (5.10) for triple integrals, Szücs derives from (5.18) a field of vectors whose curl, when suitably defined furnishes a parametric representation of the functions $u(x, y, z)$, $v(x, y, z)$, $t(x, y, z)$.

Szücs shows that if the coordinates of the vector field are assumed to have certain partial derivatives continuous, the curl taken in the classical sense is identical with that which serves as a basis for his arguments. For example, if the partial derivatives, Y_x, Z_y, X_z of a vector field $(X(x, y, z), Y(x, y, z), Z(x, y, z))$ exist and are continuous, then

$$(5.19) \quad u = Z_y - Y_z, \quad v = X_z - Z_x, \quad t = Y_x - X_y.$$

6. Schauder's proof of Haar's Theorem by means of Legendre polynomials. Schauder, 1928 (131), obtains Haar's results by using Legendre polynomials. His method is of interest since in the course of the proof the relation between Mason's lemma for $m = 1$ and Haar's theorem becomes apparent. This relationship will be indicated in § 10 below.

The functions $u(x, y)$ and $v(x, y)$ are assumed bounded and measurable, and $\mathfrak{F}(x, y)$ is a function vanishing on the boundary of B , and satisfying a Lipschitz condition in the interior of B . If B is assumed simply connected, it is sufficient to verify (5.10) when

$$(5.20) \quad \iint_G (u \mathfrak{F}_x + v \mathfrak{F}_y) dx dy = 0$$

for all $\mathfrak{F}(x, y)$ vanishing on the boundary of G , where G is square with vertices $(-1, -1), (-1, 1), (1, -1), (1, 1)$.

If

$$\mathfrak{f}_{mn}(x, y) \equiv P_m^{(x)} P_n^{(y)}, \quad (m \geq 1, n \geq 1), \quad P_n^{(x)} \equiv \int_{-1}^x P_n(x) dx$$

where $P_0(x), P_1(x), \dots, P_n(x), \dots$ are the Legendre polynomials, then each f_{mn} is an admissible $f(x,y)$ vanishing on the boundary of the square, possessing continuous derivatives and hence certainly satisfying a Lipschitz condition.

Substitution for $f(x,y)$ in (5.20) and integration by parts gives

$$(5.21) \quad \int_{-1}^1 \int_{-1}^1 \left[\int_{-1}^y u \, dy + \int_{-1}^x v \, dx \right] P_m(x) P_n(y) \, dx \, dy = 0, \quad m \geq 1, n \geq 1.$$

If

$$\phi(x,y) \equiv \int_{-1}^y u \, dy + \int_{-1}^x v \, dx$$

is expanded with respect to the orthogonal system

$\{P_m(x), P_n(y)\}$, $m \geq 0, n \geq 0$, it follows from (5.21) that

$$(5.22) \quad \phi(x,y) = f(y) + \psi(x)$$

almost everywhere. Now if

$$w(x,y) \equiv \int_{-1}^y u \, dy - \psi(x) = - \int_{-1}^x v \, dx + f(y),$$

Haar's result

$$(5.10) \quad \partial w / \partial x = -v(x,y), \quad \partial w / \partial y = u(x,y)$$

holds almost everywhere.

Schauder also proves the converse of Haar's theorem.

7. Extensions of Mason's lemma by the author of this paper. Mason's method can be extended to triple integrals to prove the following theorem.

Theorem. If $N(x, y, z)$ is continuous on $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$, $z_1 \leq z \leq z_2$ and if

$$(5.23) \quad \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} N(x, y, z) \frac{\partial^{3m} \xi}{\partial x^m \partial y^m \partial z^m} dx dy dz = 0$$

for every continuous function $\xi(x, y, z)$ such that

$$(5.24) \quad \frac{\partial^{i+j+k} \xi}{\partial x^i \partial y^j \partial z^k} \quad (i, j, k = 0, \dots, m)$$

exist and are continuous on $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$, $z_1 \leq z \leq z_2$, and such that

$$\xi(x_1, y, z) = 0, \quad \frac{\partial^i \xi(x_1, y, z)}{\partial x^i} = 0, \quad (i = 1, \dots, m-1)$$

for $y_1 \leq y \leq y_2$, $z_1 \leq z \leq z_2$;

$$\xi(x, y_1, z) = 0, \quad \frac{\partial^j \xi(x, y_1, z)}{\partial y^j} = 0, \quad (j = 1, \dots, m-1)$$

for $x_1 \leq x \leq x_2$, $z_1 \leq z \leq z_2$;

$$\xi(x, y, z_1) = 0, \quad \frac{\partial^k \xi(x, y, z_1)}{\partial z^k} = 0, \quad (k = 1, \dots, m-1)$$

for $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$; then $N(x, y, z)$

necessarily has the form

$$(5.25) \quad N = \sum_{i=0}^{m-1} [x^i f_i(y, z) + y^i g_i(x, z) + z^i h_i(x, y)] \\ + \sum_{i,k=0}^{m-1} [x^i y^k F_{ik}(z) + x^i z^k G_{ik}(y) + y^i z^k H_{ik}(x)]$$

where $f_i(y, z)$, $g_i(x, z)$, $h_i(x, y)$, $F_{ik}(z)$, $G_{ik}(y)$,

$H_{ik}(x)$ are continuous functions.

Integration by parts shows that (5.25) is sufficient for (5.23). Then¹

$$(5.26) \quad \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} (N - x^i f_i - y^i g_i - z^i h_i - x^i y^k F_{ik} - x^i z^k G_{ik} - y^i z^k H_{ik}) \frac{\partial^3 \xi}{\partial x^m \partial y^m \partial z^m} dx dy dz = 0$$

for all $\xi(x, y, z)$ satisfying the conditions of the theorem and all continuous $f_i(y, z)$, $g_i(x, z)$, $h_i(x, y)$, $F_{ik}(z)$, $G_{ik}(y)$, $H_{ik}(x)$. The function $\xi(x, y, z)$ is defined by

$\xi(x, y, z) \equiv f_{m m m}$ where

$$f_{\dots} \equiv \int_{z_1^{(m)}}^z \int_{y_1^{(m)}}^y \int_{x_1^{(m)}}^x N(x, y, z) dx^m dy^m dz^m,$$

.....

1) Repeated indices in (5.26) indicate summation from 0 to m-1.

$$f_{m \ m-1 \ m-1} \equiv f_{m-1 \ m-1 \ m-1} - \frac{(x-x_1)^m (x-x_2)^{m-1}}{(m-1)! (x_2-x_1)^m} \left[\frac{\partial^{m-1} f_{m-1 \ m-1 \ m-1}}{\partial x^{m-1}} \right]_{x=x_2}$$

$$f_{m \ m \ m-1} \equiv f_{m \ m-1 \ m-1} - \frac{(y-y_1)^m (y-y_2)^{m-1}}{(m-1)! (y_2-y_1)^m} \left[\frac{\partial^{m-1} f_{m \ m-1 \ m-1}}{\partial y^{m-1}} \right]_{y=y_2}$$

$$f_{m \ m \ m} \equiv f_{m \ m \ m-1} - \frac{(z-z_1)^m (z-z_2)^{m-1}}{(m-1)! (z_2-z_1)^m} \left[\frac{\partial^{m-1} f_{m \ m \ m-1}}{\partial z^{m-1}} \right]_{z=z_2}$$

Then $f(x, y, z) \equiv f_{m \ m \ m}(x, y, z)$ satisfies the conditions of the theorem. It is seen that

$$(5.27) \quad \frac{\partial^{3m} f(x, y, z)}{\partial x^m \partial y^m \partial z^m} = N - x^i f_i(y, z) - y^i g_i(x, z) - z^i h_i(x, y) - x^i y^k F_{ik}(z) - x^i z^k G_{ik}(y) - y^i z^k H_{ik}(x).$$

With (5.27) substituted in equation (5.26)

$$\int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} (N - x^i f_i - y^i g_i - z^i h_i - x^i y^k F_{ik} - x^i z^k G_{ik} - y^i z^k H_{ik})^2 dx dy dz = 0$$

Hence the expression in the parenthesis of the integrand must vanish, which gives the desired result (5.25).

This same method can be used to obtain the analogous theorem for a multiple integral of any order. Let $f_{\alpha_1, \alpha_2, \dots, \alpha_p}$ denote a continuous function of all the variables $x_1, x_2, \dots, x_n$

except the variables $x_{\alpha_1}, x_{\alpha_2}, \dots, x_{\alpha_p}$. Repeated subscripts below will indicate summation with respect to those subscripts from 1 to n . Mason's method can then be used to furnish the following theorem.

Theorem. If $N(x_1, x_2, \dots, x_n)$ is continuous on $a_i \leq x_i \leq b_i$ ($i = 1, \dots, n$) and if

$$\int_{a_i}^{b_i} N(x_1, x_2, \dots, x_n) \frac{\partial^{nm} \gamma}{\partial x_1^m \dots \partial x_n^m} dx_1 dx_2 \dots dx_n = 0$$

for every continuous $\gamma(x_1, x_2, \dots, x_n)$ such that

$$\frac{\partial^{i_1 + i_2 + \dots + i_n} \gamma}{\partial x_1^{i_1} \partial x_2^{i_2} \dots \partial x_n^{i_n}} (i_1, i_2, \dots, i_n = 0, \dots, m)$$

exist and are continuous on $a_i \leq x_i \leq b_i$ ($i = 1, \dots, n$) and such that

$$\gamma(x_1, x_2, x_3, \dots, x_n) = 0, \quad \frac{\partial^k \gamma(x_1, x_2, \dots, x_n)}{\partial x_1^k} = 0, \quad (k = 1, \dots, m-1)$$

for $a_j \leq x_j \leq b_j$, ($j = 2, \dots, n$), $\gamma_1 = a_1, b_1$;

$$\gamma(x_1, \gamma_2, x_3, \dots, x_n) = 0, \quad \frac{\partial^k \gamma(x_1, \gamma_2, x_3, \dots, x_n)}{\partial x_2^k} = 0, \quad (k = 1, \dots, m-1)$$

for $a_j \leq x_j \leq b_j$, ($j = 1, 3, 4, \dots, n$), $\gamma_2 = a_2, b_2$;

.....

$$\gamma(x_1, x_2, \dots, x_{n-1}, \gamma_n) = 0, \quad \frac{\partial^k \gamma(x_1, \dots, x_{n-1}, \gamma_n)}{\partial x_n^k} = 0, \quad (k = 1, \dots, m-1)$$

for $a_j \leq x_j \leq b_j$, ($j = 1, 2, \dots, n-1$), $\gamma_n = a_n, b_n$; then

$N(x_1, \dots, x_n)$ necessarily has the form

$$N(x_1, \dots, x_n) = \sum_{i=0}^{m-1} x_{\alpha_i}^i f_{\alpha_i} + \sum_{i,j=0}^{m-1} x_{\alpha_i}^i x_{\alpha_j}^j f_{\alpha_i \alpha_j} + \sum_{i,j,k=0}^{m-1} x_{\alpha_i}^i x_{\alpha_j}^j x_{\alpha_k}^k f_{\alpha_i \alpha_j \alpha_k} + \dots + \sum_{i_1, \dots, i_{n-1}=0}^{m-1} x_{\alpha_{i_1}}^{i_1} \dots x_{\alpha_{i_{n-1}}}^{i_{n-1}} f_{\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_{n-1}}}$$

It is also worth noting that Du Bois Reymond's lemma can be proved by a modification of this method. It is assumed that $N(x)$ is continuous and that equation (4-1) holds for all $\eta(x)$ vanishing with their first $m-1$ derivatives at $x=a, x=b$, having continuous m th derivatives.

For the functions $f_0, \dots, f_m$ defined by the following equations

$$\begin{aligned} f_0 &= \int_a^x N(x) dx \\ &\dots \dots \dots \\ f_m &= f_{m-1} - \frac{(x-a)^m (x-b)^{m-1}}{(m-1)! (b-a)^m} \left[\frac{\partial^{m-1} f_{m-1}}{\partial x^{m-1}} \right]_{x=b}, \end{aligned}$$

the function $\eta(x) \equiv f_m(x)$ is an admissible $\eta(x)$ and the proof goes through as in the double integral case.

The Mason method as thus extended is then very general, giving results which answer Du Bois Reymond's objection for a calculus of variations problem involving either a simple integral or a multiple integral of any order.

8. Extension of Kubota's method to triple integrals by the author. From lemma 1, 3 of Kubota and three auxiliary lemmas 2, 3, 4. that will be given in this section, the following theorem can be proved

Theorem. If $N(x, y, z)$ is continuous for
 $x_1 \leq x \leq x_2, y_1 \leq y \leq y_2, z_1 \leq z \leq z_2$ and if

for every continuous function $f(x, y, z)$ such that

$$\frac{\partial^{i+j+k} f}{\partial x^i \partial y^j \partial z^k} \quad i = 0, \dots, m; \quad j = 0, \dots, n; \quad k = 0, \dots, q)$$

exist and are continuous for $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$,

$z_1 \leq z \leq z_2$, and such that

$$f(x_k, y, z) = 0, \quad \frac{\partial^i f(x_k, y, z)}{\partial x^i} = 0, \quad (k = 1, 2; \quad i = 1, \dots, m-1),$$

for $y_1 \leq y \leq y_2$, $z_1 \leq z \leq z_2$;

$$f(x, y_n, z) = 0, \quad \frac{\partial^j f(x, y_n, z)}{\partial y^j} = 0, \quad (k = 1, 2; \quad j = 1, \dots, n-1),$$

for $x_1 \leq x \leq x_2$, $z_1 \leq z \leq z_2$;

$$f(x, y, z_k) = 0, \quad \frac{\partial^l f(x, y, z_k)}{\partial z^l} = 0, \quad (k = 1, 2; \quad l = 1, \dots, q-1)$$

for $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$, then $N(x, y, z)$ has the form

$$N(x, y, z) = \sum_{i=0}^{m-1} x^i A_i(y, z) + \sum_{j=0}^{n-1} y^j B_j(x, z) + \sum_{k=0}^{q-1} z^k C_k(x, y) \\ + \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} x^i y^j \Pi_{ij}(z) + \sum_{i=0}^{m-1} \sum_{k=0}^{q-1} x^i z^k E_{ik}(y) + \sum_{j=0}^{n-1} \sum_{k=0}^{q-1} y^j z^k G_{jk}(x),$$

where the functions involved are continuous.

Lemma 2 ^{*}. If

$$\int_{x_1}^{x_2} F(x) N(x, y, z) dx$$

is of the form

$$(5.28) \quad \int_{x_1}^{x_2} F(x) N(x, y, z) dx = \sum_{i=1}^q a_i(y) \psi_i(z) + \sum_{j=1}^n b_j(z) \phi_j(y)$$

for every continuous function $F(x)$, where the ϕ_i and ψ_i are independent of $F(x)$, then $N(x, y, z)$ is necessarily of the form:

$$\sum_{i=1}^q A_i(x, y) \psi_i(z) + \sum_{j=1}^n B_j(x, z) \phi_j(y) + \sum_{i=1}^q \sum_{j=1}^n \phi_j(y) \psi_i(z) C_{ji}(x),$$

where all the functions involved are continuous.

This can be proved in much the same way that Kubota proves Lemma 2.¹ Without restricting the generality, it can be assumed that $\phi_1(y), \dots, \phi_n(y)$ are orthogonal and unitary, and similarly for $\psi_1(z), \dots, \psi_q(z)$, i. e.,

$$\int_{y_1}^{y_2} \phi_i(y) \phi_k(y) dy = 0, \quad i \neq k; \quad \int_{y_1}^{y_2} [\phi_i(y)]^2 dy = 1;$$

$$\int_{z_1}^{z_2} \psi_i(z) \psi_k(z) dz = 0, \quad i \neq k; \quad \int_{z_1}^{z_2} [\psi_i(z)]^2 dz = 1.$$

1) See page 64 of this paper.

Consider

$$\begin{aligned}
 \Delta &\equiv \int_{x_1}^{x_2} F(x) \left[N(x, y, z) - \sum_{i=1}^q \psi_i(z) \int_{z_1}^{z_2} N \psi_i(z) dz \right. \\
 &\quad \left. - \sum_{j=1}^n \phi_j(y) \int_{y_1}^{y_2} N \phi_j(y) dy + \sum_{j=1}^n \sum_{i=1}^q \phi_j(y) \psi_i(z) \int_{y_1}^{y_2} \int_{z_1}^{z_2} N \phi_j \psi_i dy dz \right] dx \\
 &= \sum_{i=1}^q a_i(y) \psi_i(z) + \sum_{j=1}^n b_j(z) \phi_j(y) \\
 &\quad - \sum_{i=1}^q \psi_i(z) \int_{z_1}^{z_2} \psi_i(z) \left[\int_{x_1}^{x_2} N F dx \right] dz - \sum_{j=1}^n \phi_j(y) \int_{y_1}^{y_2} \phi_j(y) \left[\int_{x_1}^{x_2} F N dx \right] dy \\
 &\quad + \sum_{i=1}^q \sum_{j=1}^n \phi_j(y) \psi_i(z) \int_{y_1}^{y_2} \phi_j(y) \left[\int_{z_1}^{z_2} \psi_i(z) \left\{ \int_{x_1}^{x_2} F N dx \right\} dz \right] dy.
 \end{aligned}$$

From (5.28) and the orthogonal and unitary properties of the functions it follows that

$$\begin{aligned}
 \Delta &= \sum_{i=1}^q a_i(y) \psi_i(z) + \sum_{j=1}^n b_j(z) \phi_j(y) \\
 &\quad - \sum_{i=1}^q \psi_i(z) \int_{z_1}^{z_2} \psi_i(z) \left[\sum_{i=1}^q a_i(y) \psi_i(z) + \sum_{j=1}^n b_j(z) \phi_j(y) \right] dz \\
 &\quad - \sum_{j=1}^n \phi_j(y) \int_{y_1}^{y_2} \phi_j(y) \left[\sum_{i=1}^q a_i(y) \psi_i(z) + \sum_{j=1}^n b_j(z) \phi_j(y) \right] dy \\
 &\quad + \sum_{i=1}^q \sum_{j=1}^n \phi_j(y) \psi_i(z) \int_{y_1}^{y_2} \phi_j(y) \left[\int_{z_1}^{z_2} \psi_i(z) \left\{ \sum_{k=1}^q a_k(y) \psi_k(z) \right. \right. \\
 &\quad \left. \left. + \sum_{h=1}^n b_h(z) \phi_h(y) \right\} dz \right] dy = 0.
 \end{aligned}$$

The function Δ vanishes then for every continuous function $F(x)$. It follows that

$$\begin{aligned} N(x, y, z) &= \sum_{i=1}^q \psi_i(z) \int_{z_1}^{z_2} N(x, y, z) \psi_i(z) dz \\ &+ \sum_{j=1}^n \phi_j(y) \int_{y_1}^{y_2} N(x, y, z) \phi_j(y) dy - \sum_{j=1}^n \sum_{i=1}^q \phi_j(y) \psi_i(z) \int_{y_1}^{y_2} \int_{z_1}^{z_2} N(x, y, z) \phi_j \psi_i dy dz \\ &\equiv \sum_{i=1}^q \psi_i(z) A_i(x, y) + \sum_{j=1}^n \phi_j(y) B_j(x, z) + \sum_{j=1}^n \sum_{i=1}^q \phi_j(y) \psi_i(z) C_{ji}(x). \end{aligned}$$

Lemma 3^m. If

$$\int_{x_1}^{x_2} F(x) N(x, y, z) dx = \sum_{i=1}^q a_i(y) \psi_i(z) + \sum_{j=1}^n b_j(z) \phi_j(y)$$

for every continuous function $F(x)$ such that

$$\int_{x_1}^{x_2} F(x) f_i(x) dx = 0, \quad i = 1, \dots, m,$$

where $\phi_j(y)$ and $\psi_i(z)$ are independent of $F(x)$, then $N(x, y, z)$ necessarily has the form

$$\begin{aligned} N(x, y, z) &= \sum_{i=1}^q A_i(x, y) \psi_i(z) + \sum_{j=1}^n B_j(x, z) \phi_j(y) \\ &+ \sum_{j=1}^n \sum_{i=1}^q \phi_j(y) \psi_i(z) C_{ji}(x) + \sum_{h=1}^m \Pi_h(y, z) f_h(x), \end{aligned}$$

where all the functions involved are continuous.

Let $F_0(x)$, $F_1(x)$, ..., $F_m(x)$ be m arbitrary continuous functions such that the determinant.

$$(5.29) \quad \left| \int_{x_1}^{x_2} F_i(x) f_k(x) dx \right| \neq 0, \quad (i, k = 1, 2, \dots, m),$$

and let

$$F(x) = F_0(x) + \varepsilon_1 F_1(x) + \dots + \varepsilon_m F_m(x),$$

the constants $\varepsilon_1, \dots, \varepsilon_m$ being determined so that

$$\int_{x_1}^{x_2} F_0(x) f_i(x) dx + \sum_{j=1}^m \varepsilon_j \int_{x_1}^{x_2} F_j(x) f_i(x) dx = 0, \quad (i = 1, \dots, m).$$

Then

$$\int_{x_1}^{x_2} N(x, y, z) F_0(x) dx + \sum_{j=1}^m \varepsilon_j \int_{x_1}^{x_2} N(x, y, z) F_j(x) dx$$

is of the form

$$\sum_{i=1}^q a_i(y) \psi_i(z) + \sum_{j=1}^n b_j(z) \phi_j(y).$$

The functions $\Pi_k(y, z)$ are determined so that

$$\int_{x_1}^{x_2} F_j(x) N(x, y, z) dx - \sum_{k=1}^m \Pi_k(y, z) \int_{x_1}^{x_2} F_k(x) f_j(x) dx = 0,$$

$$(j = 1, \dots, m).$$

The $\Pi_k(y, z)$ are uniquely determined because of the condition (5.29). It follows that

$$\int_{x_1}^{x_2} F_0(x) \left[N(x, y, z) - \sum_{k=1}^m \Pi_k(y, z) f_k(x) \right] dx$$

has the form

$$\sum_{i=1}^q a_i(y) \psi_i(z) + \sum_{j=1}^n b_j(z) \phi_j(y)$$

for every continuous function $F_0(x)$. From Lemma 2*,

$$N - \sum_{k=1}^m \Pi_k(y, z) f_k(x) = \sum_{i=1}^q A_i(x, y) \psi_i(z) + \sum_{j=1}^n B_j(x, z) \phi_j(y) + \sum_{j=1}^n \sum_{i=1}^q \phi_j(y) \psi_i(z) C_{ji}(x),$$

and $N(x, y, z)$ has the desired form. This proof is exactly like Kubota's method of proof of Lemma 3.

Kubota's method is also followed in proving the following lemma.

Lemma 4*. If

$$(5.30) \quad \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} N(x, y, z) \xi(x, y, z) dx dy dz = 0$$

for every continuous function $\xi(x, y, z)$ such that

$$(5.31) \quad \int_{x_1}^{x_2} \xi(x, y, z) f_i(x) dx = 0, \quad (i = 1, \dots, m)$$

$$(5.32) \quad \int_{y_1}^{y_2} \xi(x, y, z) \phi_i(y) dy = 0, \quad (i = 1, \dots, n)$$

$$(5.33) \quad \int_{z_1}^{z_2} f(x, y, z) \psi_i(z) dz = 0, \quad (i = 1, \dots, q),$$

then $N(x, y, z)$ has the form

$$\begin{aligned} N(x, y, z) = & \sum_{i=1}^q A_i(x, y) \psi_i(z) + \sum_{j=1}^n B_j(x, z) \phi_j(y) \\ & + \sum_{k=1}^m C_k(y, z) f_k(x) + \sum_{j=1}^n \sum_{i=1}^q \phi_j(y) \psi_i(z) D_{ji}(x) \\ & + \sum_{k=1}^m \sum_{j=1}^n f_k(x) \phi_j(y) E_{kj}(z) + \sum_{k=1}^m \sum_{i=1}^q f_k(x) \psi_i(z) C_{ki}(y), \end{aligned}$$

where all the functions involved are continuous.

Consider three continuous functions $F(x)$, $\Phi(y)$, $\Psi(z)$ such that

$$(5.34) \quad \int_{x_1}^{x_2} F f_i dx = 0, \quad (i = 1, \dots, m)$$

$$(5.35) \quad \int_{y_1}^{y_2} \Phi \phi_i dy = 0, \quad (i = 1, \dots, n)$$

$$(5.36) \quad \int_{z_1}^{z_2} \Psi \psi_i dz = 0, \quad (i = 1, \dots, q)$$

For $f(x, y, z) \equiv F(x) \Phi(y) \Psi(z)$, conditions

(5.31), (5.32), (5.33) are satisfied and hence

$$\int_{y_1}^{y_2} \int_{x_1}^{x_2} \int_{z_1}^{z_2} F(x) \Phi(y) \Psi(z) N(x, y, z) dy dx dz = 0.$$

If $F(x)$ and $\Phi(y)$ are fixed and $\Psi(z)$ is arbitrary, Lemma 1 of Kubota says that

$$\int_{y_1}^{y_2} \int_{x_1}^{x_2} \Phi(y) F(x) N(x, y, z) dx dy$$

is of the form $C_1 \psi_1(z) + \dots + C_q \psi_q(z)$. If now $F(x)$ is fixed and $\Phi(y)$ is allowed to vary, Lemma 3 of Kubota says that

$$\int_{x_1}^{x_2} F(x) N(x, y, z) dx$$

is of the form

$$\sum_{i=1}^q c_i(y) \psi_i(z) + \sum_{j=1}^n b_j(z) \phi_j(y)$$

Then by Lemma 3,* $N(x, y, z)$ has the form

$$(5.37) \quad N(x, y, z) = \sum_{i=1}^q A_i^1(x, y) \psi_i(z) + \sum_{j=1}^n B_j^1(x, z) \phi_j(y) \\ + \sum_{j=1}^n \sum_{i=1}^q \phi_j(y) \psi_i(z) C_{ji}^1(x) + \sum_{h=1}^m \Pi_h^1(y, z) f_h(x).$$

Since the functions involved are continuous, this result should be independent of the order of integration. If the foregoing procedure is repeated, first by letting the functions vary in the order F, Φ, Ψ , then in the order Ψ, F, Φ , the following results are obtained

$$(5.38) \quad N(x, y, z) = \sum_{h=1}^m \Pi_h^2(y, z) f_h(x) + \sum_{j=1}^n B_j^2(x, z) \phi_j(y) \\ + \sum_{h=1}^m \sum_{j=1}^n G_{hj}^2(z) f_h(x) \phi_j(y) + \sum_{i=1}^q A_i^2(x, y) \psi_i(z), \\ N(x, y, z) = \sum_{i=1}^q A_i^3(x, y) \psi_i(z) + \sum_{h=1}^m \Pi_h^3(y, z) f_h(x) \\ + \sum_{i=1}^q \sum_{h=1}^m H_{ih}^3(y) \psi_i(z) f_h(x) + \sum_{j=1}^n B_j^3(x, z) \phi_j(y).$$

(5.39)

$$\begin{aligned}
 N(x, y, z) = & \sum_{i=1}^q A_i^3(x, y) \psi_i(z) + \sum_{h=1}^m \Pi_h^3(y, z) f_h(x) \\
 & + \sum_{i=1}^q \sum_{h=1}^m H_{ih}^3(y) \psi_i(z) f_h(x) + \sum_{j=1}^n B_j^3(x, z) \phi_j(y).
 \end{aligned}$$

The three results (5.37), (5.38), (5.39) say that N is of the form stated in the lemma.

The theorem of page 78 can be proved from Lemma 4* in the same way that the theorem for double integrals was proved from Lemma 4. If

$$f(x, y, z) = \int_{z_1}^z \dots \int_{z_1}^z \int_{y_1}^y \dots \int_{y_1}^y \int_{x_1}^x \dots \int_{x_1}^x f(x, y, z) dx^m dy^n dz^q,$$

the conditions imposed on f at x_1, y_1, z_1 are obviously satisfied and the conditions at x_2, y_2, z_2 are satisfied if

$$\int_{x_1}^{x_2} x^i f(x, y, z) dx = 0 \quad (i = 0, \dots, m-1),$$

$$\int_{y_1}^{y_2} y^i f(x, y, z) dy = 0 \quad (i = 0, \dots, n-1),$$

$$\int_{z_1}^{z_2} z^i f(x, y, z) dz = 0 \quad (i = 0, \dots, q-1).$$

The theorem of page 78 follows then immediately from Lemma 4*.

9. Extension of Schauder's method to triple integrals by the author. Legendre polynomials can be used to obtain the result analogous to (5.22) for triple integrals. If

$$(5.40) \quad \alpha = \iiint_G [u(x, y, z) f_x + v(x, y, z) f_y + t(x, y, z) f_z] dx dy dz,$$

where G is a cube with vertices $(\pm 1, \pm 1, \pm 1)$ and

$f(x, y, z)$ is any function vanishing on the boundary of

G , having continuous first partial derivatives then there

exist continuous functions $f_1(x, y)$, $f_2(x, z)$, $f_3(y, z)$,

$f_4(x)$, $f_5(y)$, $f_6(z)$ such that

$$(5.41) \quad \int_{-1}^z \int_{-1}^y u dz dy + \int_{-1}^x \int_{-1}^z v dx dz + \int_{-1}^y \int_{-1}^x t dx dy =$$

$$f_1(x, y) + f_2(x, z) + f_3(y, z) + f_4(x) + f_5(y) + f_6(z).$$

To prove this, define the following functions:

$$Y_{m n q} \equiv P_m(x) P_n(y) P_q(z), \quad m \geq 1, n \geq 1, q \geq 1,$$

$$P_m(x) \equiv \int_{-1}^x P_m(x) dx,$$

where P_m ($m \geq 0$) are Legendre polynomials. Then $\int_m n q$ are admissible $\gamma(x, y, z)$ for $m, n, q \geq 1$. With these functions substituted in (5.40), integration by parts gives

$$(5.42) \quad \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 P_m(x) P_n(y) P_q(z) \left[\int_{-1}^z \int_{-1}^y u \, dy \, dz + \int_{-1}^x \int_{-1}^z v \, dx \, dz + \int_{-1}^x \int_{-1}^y t \, dx \, dy \right] dx \, dy \, dz = 0, \quad (m, n, q \geq 1).$$

If

$$\phi(x, y, z) \equiv \int_{-1}^z \int_{-1}^y u \, dy \, dz + \int_{-1}^x \int_{-1}^z v \, dx \, dz + \int_{-1}^x \int_{-1}^y t \, dx \, dy$$

is expanded with respect to the system $\{P_m(x), P_n(y), P_l(z)\}$

$$\begin{aligned} \phi(x, y, z) &= \sum_{i,j,k=0}^{\infty} C_{ijk} \equiv \sum_{k=0}^{\infty} C_{00k} + \sum_{j=1}^{\infty} C_{0j0} \\ &+ \sum_{j,k=1}^{\infty} C_{0jk} + \sum_{i=1}^{\infty} C_{i00} + \sum_{i,k=1}^{\infty} C_{i0k} + \sum_{i,j=1}^{\infty} C_{ijo} + \sum_{i,j,k=1}^{\infty} C_{ijk}, \end{aligned}$$

where

$$C_{ijk} \equiv \Pi_{ijk} P_i(x) P_j(y) P_k(z),$$

the Π_{ijk} being constants. It follows from (5.42) that

$\Pi_{ijk} = 0$ for $i \geq 1, j \geq 1, k \geq 1$. Hence if u, v, t are bounded and measurable,

$$\phi(x, y, z) = f_1(x, y) + f_2(x, z) + f_3(y, z) + f_4(x) + f_5(y) + f_6(z)$$

almost everywhere.

10. Application of the various lemmas to the calculus of variations.

The relations between the results given by the various lemmas for a calculus of variations problem will now be investigated. Problems where first partial derivatives only occur in the integrand will be considered, not merely for the sake of simplicity but because Haar's theorem applies to such problems only.

If the second partial derivatives of z exist and are continuous, then by the Lagrange integration by parts, the vanishing of the first variation of J (5.1) given by equation (5.2) can be expressed in the form (5.3) where

$$(5.43) \quad M = f_z - \frac{\partial}{\partial x} f_p - \frac{\partial}{\partial y} f_q = 0.$$

In order to apply Mason's lemma it is sufficient to assume that only the first partial derivatives of z exist and are continuous. Admissible functions $\zeta(x,y)$ for which equation (5.2) holds are functions vanishing on the boundary of B such that

$$(5.44) \quad \frac{\partial^{i+j} \zeta(x,y)}{\partial x^i \partial y^j}, \quad (i, j = 0, 1)$$

exist and are continuous. However, functions $\zeta(x, y)$ defined in some rectangle R in the interior of B with vertices (a,b) , (a,β) , (α,b) , (α,β) having the same continuity properties, vanishing on the boundary of the rectangle, and defined to be identically zero in the part

of B outside R are also admissible $\zeta(x, y)$. Then for all such $\zeta(x, y)$

$$(5.45) \quad \int_a^\alpha \int_b^\beta (f_z \zeta + f_p \zeta_x + f_q \zeta_y) dx dy = 0.$$

The relations $\zeta(x, b) = \zeta(x, \beta) = \zeta(a, y) = \zeta(\alpha, y) = 0$ and (5.44) imply $\zeta_x(x, b) = \zeta_x(x, \beta) = \zeta_y(a, y) = \zeta_y(\alpha, y) = 0$. With these conditions and integration by parts equation (5.45) becomes

$$(5.46) \quad \int_a^\alpha \int_b^\beta N(x, y) \zeta_{xy} dx dy = 0,$$

where

$$N \equiv \int_a^x \int_b^y f_z dx dy - \int_b^y f_p dy - \int_a^x f_q dx.$$

Mason's lemma of page 56 then says that

$$(5.47) \quad N = Y(y) + X(x)$$

interior to R . But R was an arbitrary rectangle in B and hence if B is simply connected, this relation (5.47) holds in B .

If it is assumed that second partial derivatives of z exist, differentiation of equation (5.47) with respect to x and y gives the Lagrange equation (5.43).

With the following notation introduced for Ω, u, v ,

$$2\Omega \equiv \int_b^y \int_a^x f_z dx dy,$$

$$u \equiv \Omega_y - f_p, \quad v \equiv \Omega_x - f_q,$$

the function N can be expressed by the formula

$$(5.48) \quad N \equiv \int_b^y u \, dy + \int_a^x v \, dx = Y(y) + X(x).$$

But this is the result (5.22) obtained by Schauder in the course of his proof of Haar's theorem, from which he obtains relations (5.10) which are equivalent to (5.7).

For double integrals, then, Haar's theorem seems to give nothing new since the same results can be obtained from Mason's lemma.

Now consider the problem of minimizing

$$\iiint_T f(x, y, z, w, p, q, r) \, dx \, dy \, dz,$$

where $p = \partial w / \partial x$, $q = \partial w / \partial y$, $r = \partial w / \partial z$

in the class of hypersurfaces $w = w(x, y, z)$ whose projection on the ordinary space xyz is T . The condition that the first variation vanish is expressed by the equation

$$(5.49) \quad \iiint_T (f_w \zeta + f_p \zeta_x + f_q \zeta_y + f_r \zeta_z) \, dx \, dy \, dz = 0,$$

which must hold for all variations $\zeta(x, y, z)$ which vanish on the boundary of T and have continuous first partial derivatives in T .

By the Lagrange integration by parts this equation (5.49) can be expressed in the form

$$\iiint_T (f_w - \frac{\partial}{\partial x} f_p - \frac{\partial}{\partial y} f_q - \frac{\partial}{\partial z} f_r) \zeta \, dx \, dy \, dz = 0.$$

For this integration by parts it is sufficient to assume that the second partial derivatives of w exist and are continuous. By an argument similar to Kneser's (121) for double integrals, it follows that

$$(5.50) \quad f_w - \frac{\partial}{\partial x} f_p - \frac{\partial}{\partial y} f_q - \frac{\partial}{\partial z} f_r = 0.$$

If (5.49) is assumed to hold for all $\gamma(x, y, z)$ vanishing on the boundary of T , for which

$$\frac{\partial^{i+j+k} \gamma}{\partial x^i \partial y^j \partial z^k} \quad (i, j, k = 0, 1)$$

all exist and are continuous, a transformation of equation (5.49) can be made so that the analogue of Mason's lemma can be applied. For this purpose, it is sufficient to assume that the first partial derivatives of w are continuous. As in the problem for double integrals, functions $\gamma(x, y, z)$ with the same continuity properties, defined in some rectangular parallelepiped R interior to T with vertices (x_i, y_k, z_j) , $(i, j, k = 1, 2)$ and vanishing on the boundary of R , identically zero in the part of T outside R , are also admissible functions $\gamma(x, y, z)$. Hence for all such $\gamma(x, y, z)$, equation (5.49) holds with T replaced by R . Transformation of this equation by integration by parts as in the double integral problem gives

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} N(x, y, z) \frac{\partial^3 \gamma}{\partial x \partial y \partial z} dx dy dz = 0,$$

where

$$N(x, y, z) \equiv \int_{y_1}^y \int_{z_1}^z f_p dy dz + \int_{x_1}^x \int_{z_1}^z f_q dx dz + \int_{x_1}^x \int_{y_1}^y f_r dx dy - \int_{x_1}^x \int_{y_1}^y \int_{z_1}^z f_w dx dy dz.$$

By the analogue of Mason's lemma on page 105

$$(5.51) \quad N = f_1(x, y) + f_2(x, z) + f_3(y, z) + f_4(x) + f_5(y) + f_6(z)$$

in R , and if T is simply connected, in T . If the second partial derivatives are assumed to exist, differentiation of N with respect to x , y , and z gives the Lagrange equation (5.50)

The analogue of (5.48) can be obtained by giving

N the form

$$(5.52) \quad N \equiv \int_{z_1}^z \int_{y_1}^y u(x, y, z) dy dz + \int_{z_1}^z \int_{x_1}^x v(x, y, z) dx dz + \int_{x_1}^x \int_{y_1}^y t(x, y, z) dx dy,$$

where

$$u \equiv f_p - \Omega_{yz}, \quad v \equiv f_q - \Omega_{xz}, \quad t \equiv f_r - \Omega_{xy},$$

$$3 \Omega \equiv \int_{x_1}^x \int_{y_1}^y \int_{z_1}^z f_w dx dy dz.$$

with N expressed as in (5.52), by means of equation (5.51), a result is obtained which agrees with (5.41) obtained in § 9. This is the analogue of equation (5.48).

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VITA

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